

Minimal Set of States for Optimizing Quantum Gates in Open Quantum Systems

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“Optimal Control of Quantum Systems”

$$\text{CPHASE} = \text{diag}(-1, 1, 1, 1)$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Standard approach to quantum gate optimization

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Goal: Maximize

$$F = \frac{1}{d} \sum_{i=1}^d \Re \langle \psi_i | \hat{\mathbf{O}}^\dagger \hat{\mathbf{P}} \hat{\mathbf{U}}(T, 0, \epsilon) \hat{\mathbf{P}} | \psi_i \rangle$$

Two-qubit gates: $d = 4$

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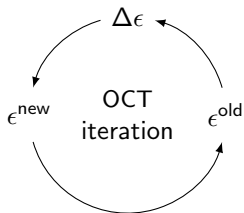
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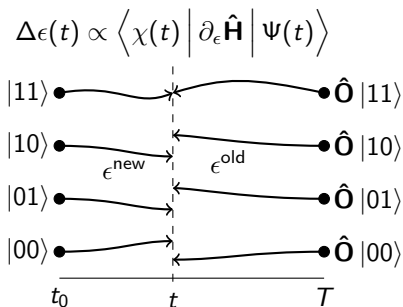
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In the real world: decoherence

$$\hat{\rho}(T) = \mathcal{D}(\hat{\rho}(0)); \quad \text{e.g. } \frac{\partial \hat{\rho}}{\partial t} = \frac{i}{\hbar} [\hat{\mathbf{H}}, \hat{\rho}] + \mathcal{L}_D(\hat{\rho})$$

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Lift $F = \frac{1}{d} \sum_{i=1}^d \Re \langle \Psi_i | \hat{\mathbf{O}}^\dagger \hat{\mathbf{P}} \hat{\mathbf{U}}(T, 0, \epsilon) \hat{\mathbf{P}} | \Psi_i \rangle$ to Liouville space.

Kallush & Kosloff, Phys. Rev. A 73, 032324 (2006),

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Schulte-Herbrüggen et al., J. Phys. B 44, 154013 (2011)

$$\Rightarrow F = \frac{1}{d^2} \sum_{j=1}^{d^2} \text{tr} \left[\hat{\mathbf{O}} \hat{\rho}_j(0) \hat{\mathbf{O}}^\dagger \hat{\rho}_j(T) \right]$$

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$$\hat{\rho}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\rho}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\rho}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \dots$$

d^2 matrices to propagate! (16 for two-qubit gate)

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Claim

We only need to propagate **three** matrices (independent of d), instead of d^2 .

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E.g. $\hat{\mathbf{O}} = \text{diag}(-1, 1, 1, 1)$;

For $\hat{\mathbf{U}} = \mathbb{1}$

using just $\hat{\rho}_1$ will not distinguish $\hat{\mathbf{U}}$ from $\hat{\mathbf{O}}$. ($\hat{\mathbf{U}}\hat{\rho}_1\hat{\mathbf{U}}^\dagger = \hat{\mathbf{O}}\hat{\rho}_1\hat{\mathbf{O}}^\dagger$)

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$\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3$ together guarantee that $\mathcal{D}(\hat{\rho})$ is unitary on the logical subspace.

A reduced set of density matrices

$$\blacksquare \hat{\rho}_3 = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

dynamical map in the logical subspace

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Totally rotated state: relative phases between mapped logical eigenstates

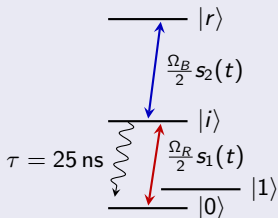
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An example: Optimization of a Rydberg Gate

Two trapped neutral atoms

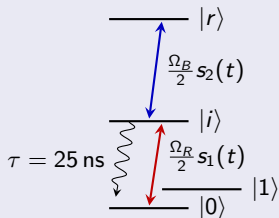
Single-qubit Hamiltonian



$$\hat{H}_{1q} = \begin{pmatrix} 0 & 0 & \frac{\Omega_R}{2} s_1(t) & 0 \\ 0 & E1 & 0 & 0 \\ \frac{\Omega_R}{2} s_1(t) & 0 & \Delta_1 & \frac{\Omega_B}{2} s_2(t) \\ 0 & 0 & \frac{\Omega_B}{2} s_2(t) & \Delta_2 \end{pmatrix}$$

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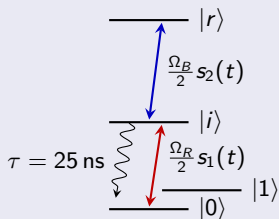
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$$\hat{H}_{2q} = \hat{H}_{1q} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_{1q} - U |rr\rangle \langle rr|$$

dipole-dipole interaction when both atoms in Rydberg state

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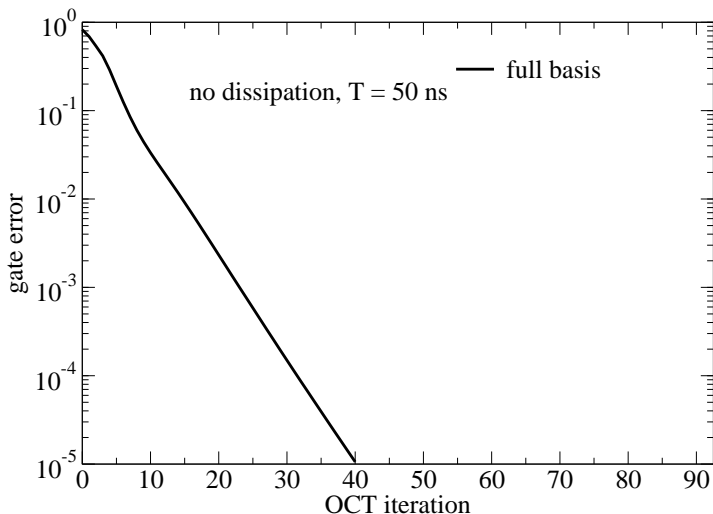
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no coupling between $|0\rangle$, $|1\rangle \Rightarrow$ only diagonal gates

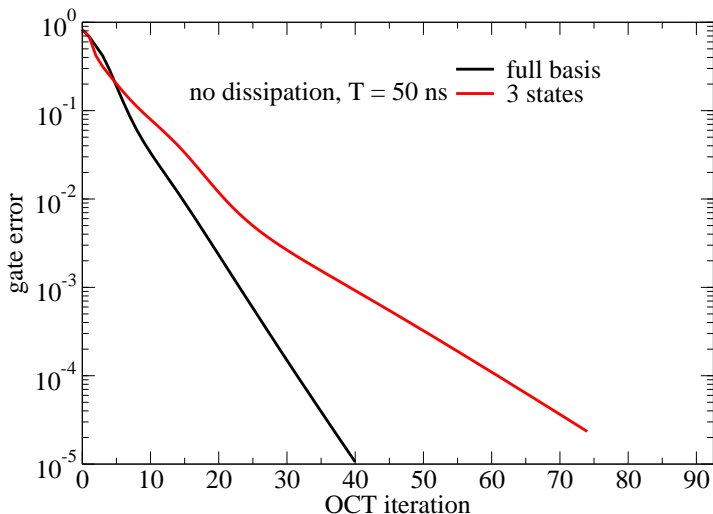
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first: optimize in Liouville space
– but without dissipation

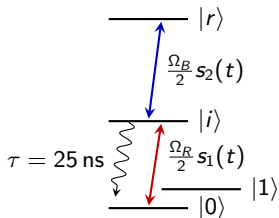
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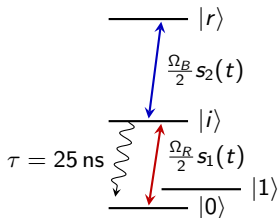


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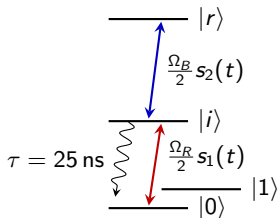
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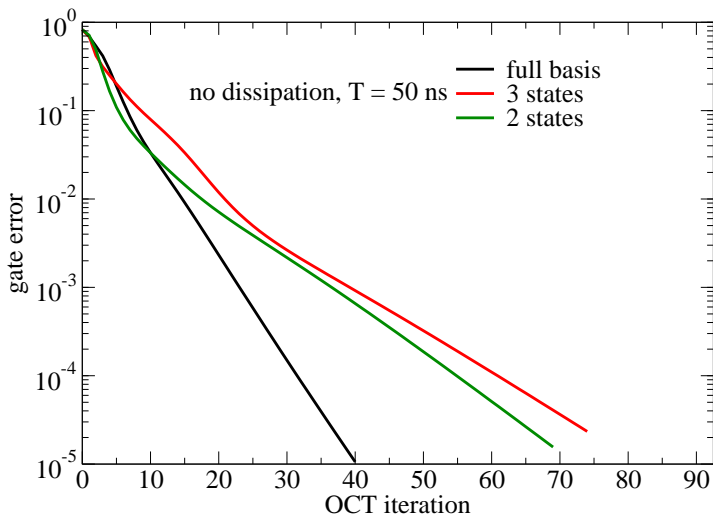
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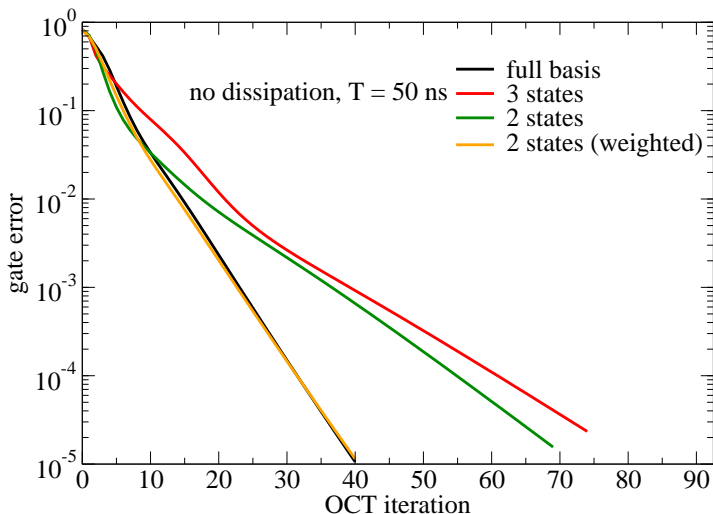
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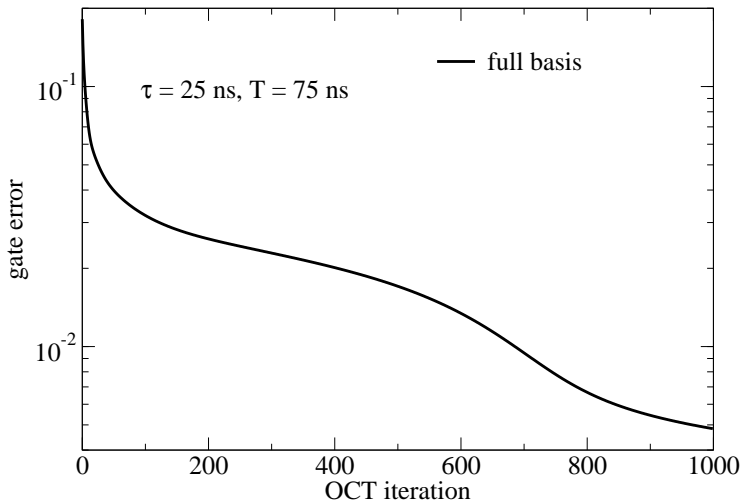


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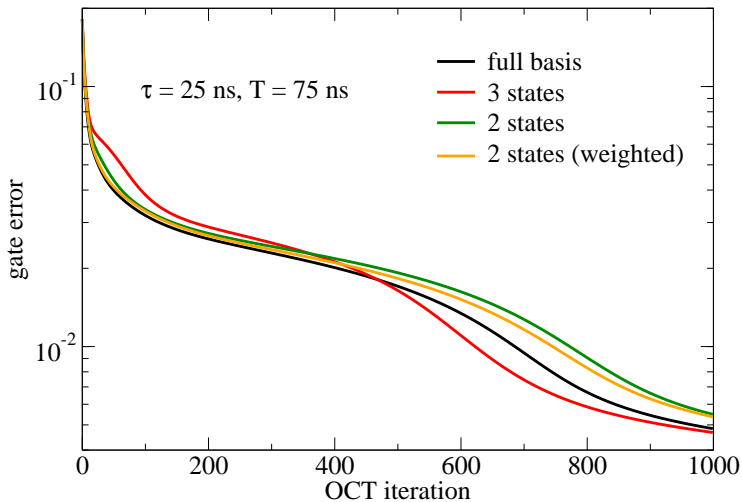


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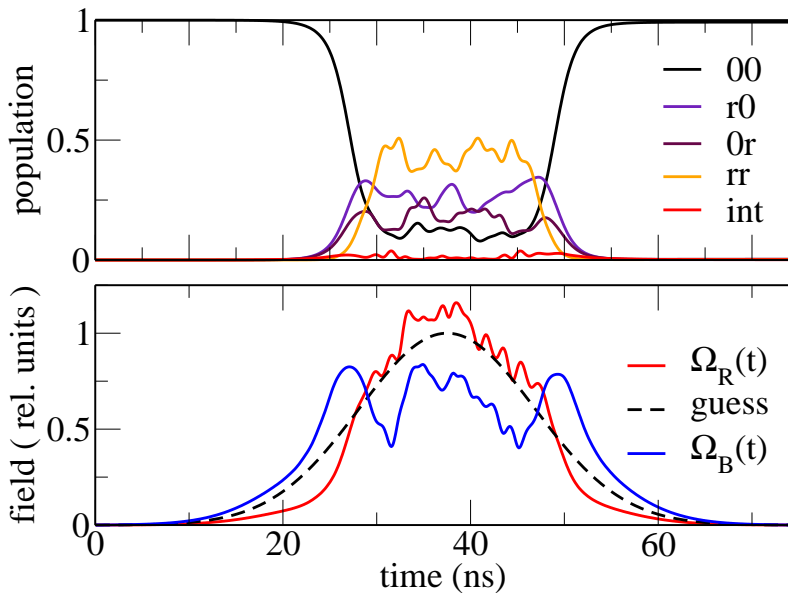
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Optimized dynamics



- A set of three density matrices is sufficient for gate optimization: (independent of dimension of Hilbert space!)
 - one to check dynamical map on subspace
 - one to check the basis
 - one to check the phases
- Further reduction possible for restricted systems
- States can be weighted according to physical interpretation

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To do:

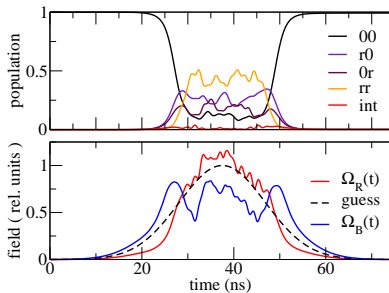
- Test with more complex Hamiltonians allowing non-diagonal gates (Transmon gates)

Thank You!

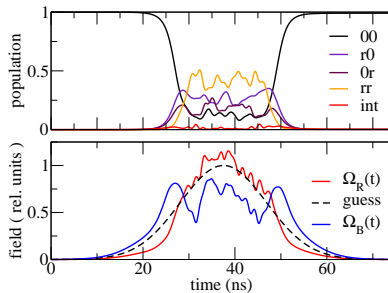
Thank You!

Reich, Gualdi, Koch. arXiv:1305.3222 (2013)

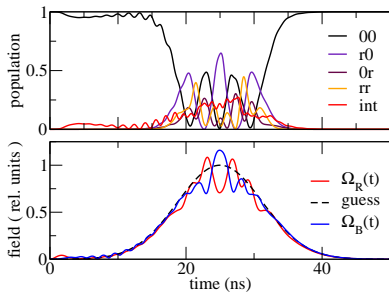
with dissipation, full basis



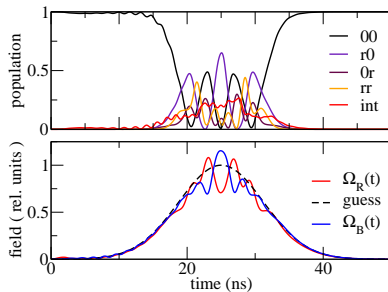
with dissipation, two states (weighted)



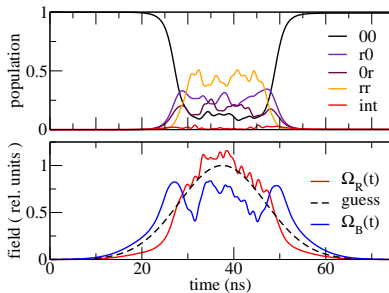
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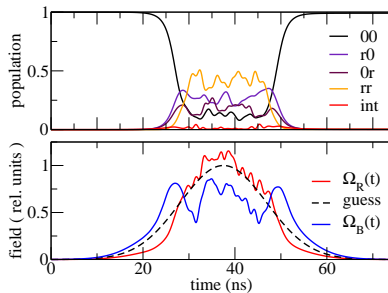
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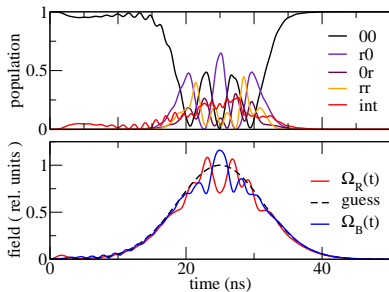
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with dissipation, two states (weighted)



without dissipation, full basis



without dissipation, three states

