Coupled Oscillations

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Introduction:

Coupled oscillations can appear in many different versions. The most common cases in physics are the coupled oscillatory circuit, the coupled spring pendulum and the coupled gravity pendulum. This experiment refers to the latter. Although the general mathematical description is almost the same, some details have to be considered.

In comparison to the single pendulum, we get an additional term in the mathematical ~~described~~ description from the coupling. Therefore we get a differential equation for each pendulum in dependency of the angle of the other pendulum:

$$\ddot{\phi}_1 + \frac{m \cdot g \cdot s}{I} \phi_1 + \frac{D r^2}{I} (\phi_1 - \phi_2) = 0$$

$$\ddot{\phi}_2 + \frac{m \cdot g \cdot s}{I} \phi_2 + \frac{D r^2}{I} (\phi_2 - \phi_1) = 0$$

D beeing the spring constant of the coupling spring and r ~~the~~ beeing the distance between pivotal point and coupling point (in symmetric case). In analogy to the single pendulum, we get the frequency ω_0 as:

$$\omega_0^2 = \frac{m \cdot g \cdot s}{I}$$

f is a quantity to describe the coupling of the pendulums:

$$f = \frac{D r^2}{m \cdot g \cdot s}$$

Now the equations can be written as follows:

$$\ddot{\phi}_1 + (1+f)\,\omega_0^2\,\phi_1 - f\,\omega_0^2\,\phi_2 = 0$$

$$\ddot{\phi}_2 - f\,\omega_0^2\,\phi_1 + (1+f)\,\omega_0^2\,\phi_2 = 0$$

This system of differential equations can be solved using the matrix form:

$$\ddot{\phi} + \underbrace{\begin{pmatrix} (1+f)\omega_0^2 & -f\omega_0^2 \\ -f\omega_0^2 & (1+f)\omega_0^2 \end{pmatrix}}_{\Omega} \phi = 0$$

As ansatz we choose: $\phi(t) = \vec{z} \cdot e^{i\omega t}$

$$\Rightarrow \quad \ddot{\phi}(t) = -\omega^2 \vec{z}\, e^{i\omega t}$$

$$\Rightarrow \quad -\omega^2 \vec{z} e^{i\omega t} + \Omega \vec{z} e^{i\omega t} = 0$$

$$\Leftrightarrow \quad (-\omega^2 \mathbb{1} + \Omega)\, \vec{z} e^{i\omega t} = 0$$

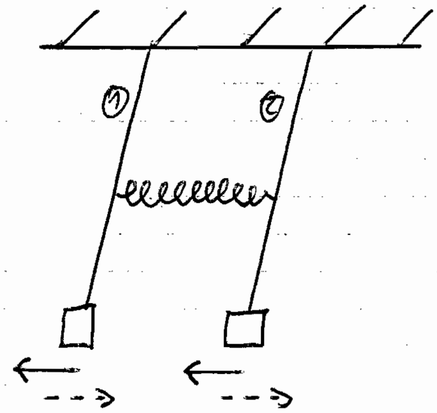
$$\Leftrightarrow \quad \underbrace{(\Omega - \omega^2 \mathbb{1})}_{M}\, \vec{z} = 0$$

The theory says, that a non-trivial solution can only be finded, if the determinant of M disappears. From this we get the four eigenvalues ω_j as the solutions

$$\omega_{1/2} = \pm \omega_0 \quad ; \quad \omega_{3/4} = \pm \omega_0 \sqrt{1+2f}$$

This are the two natural oscillations of the system:

The first one is independet from f:

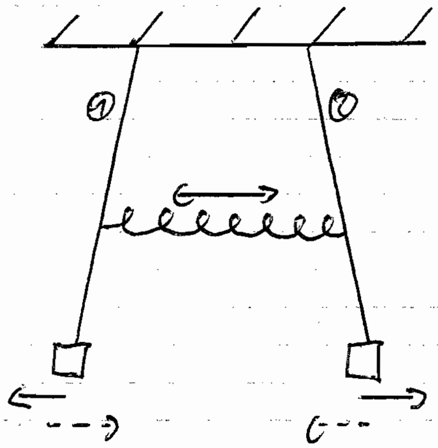


$\omega_s = \omega_0$

$A_① = A_②$

The coupling spring will not be displaced and is without any influence.

The second one is influenced by the spring and therefore dependent on f:



$\omega_a = \omega_0 \sqrt{1+2f}$

$A_① = - A_②$

If only one of the pendulums is displaced and the otherone is idle an effect called "beat" is possible: This means an energy transfer between the pendulums. The amplitude of each pendulum (maximum aplitude) will get a time dependency, while $A_①(t)$ and $A_②(t)$ will have a phase shifting.

The mathematical description is:

$$\phi_1(t) = t \cdot \phi_0 \cdot \cos \Delta\omega \cdot \cos \omega t = A_\Theta(t) \cdot \cos \omega t$$

$$\phi_2(t) = t \cdot \phi_0 \cdot \sin \Delta\omega \cdot \sin \omega t = A_\Theta(t) \sin \omega t$$

ω beeing the "sub frequency" $\omega = \frac{\omega_a + \omega_s}{2}$
and $\Delta\omega$ the periodic change of ~~am~~
maximum amplitude $\Delta\omega = \frac{\omega_a - \omega_s}{2}$

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Experiment 23. 3. 05

Used devices:

pendulum: $\Delta l = (100{,}0 \pm 0{,}1)$ mm $= \Delta r$

magnetic and gravity measurement systems

Assignment 1:

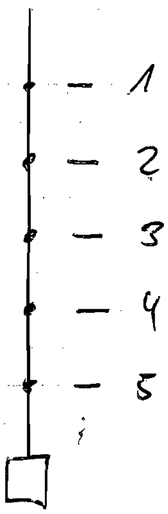
which one corresponds to g?

left pendulum: 11,59 s for 6 periods
right pendulum: 11,56 s for 6 periods
} ± 0,05 s

This values can be called identical within error. However, the measurement of the left pendulum has a higher accuracy!

Assignment 2:

Coupling point numbering:



Assignment 4:

mass: (50,0 ± 0,5) g
displacement: 15 cm

periods	time
10	8,53 s
20	17,10 s
30	25,58 s

distance with and without mass: 16,5 cm

Additional derivation

The solution of the system of differential equations in the introduction can be done via a eigenvalue equation.

$$\ddot{\phi}_1 + (1+f)\omega_0^2 \phi_1 - f\omega_0^2 \phi_2 = 0$$
$$\ddot{\phi}_2 + (1+f)\omega_0^2 \phi_2 - f\omega_0^2 \phi_1 = 0$$

First of all the equations can be written in ~~Matrix~~- or vector form:

$$\ddot{\phi} + \Omega \phi = 0$$

while $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\Omega = \begin{pmatrix} (1+f)\omega_0^2 & -f\omega_0^2 \\ -f\omega_0^2 & (1+f)\omega_0^2 \end{pmatrix}$

As described into the introduction we get the following equation by using an exponential ansatz:

(1) $$(\underbrace{\Omega - \omega^2 \mathbb{1}}_{M}) \vec{Z} = 0$$

This is a eigenvalue equation which has a non-trivial solution if the determinant of M disappears:

$$|(\Omega - \omega^2 \mathbb{1})| \stackrel{!}{=} 0$$

$$\Rightarrow \begin{vmatrix} (1+f)\omega_0^2 - \omega^2 & -(f\omega_0^2) \\ -f\omega_0^2 & (1+f)\omega_0^2 - \omega^2 \end{vmatrix} \stackrel{!}{=} 0$$

From this follows:

$$[(1+f)\omega_0^2 - \omega^2][(1+f)\omega_0^2 - \omega^2] - f^2\omega_0^4 \overset{!}{=} 0$$

$$\Leftrightarrow \quad (1+f)^2\omega_0^4 - 2(1+f)\omega_0^2\omega^2 + \omega^4 - f^2\omega_0^4 = 0$$

$$\Leftrightarrow \quad \omega^4 - 2(1+f)\omega_0^2\omega^2 + ((1+f)^2\omega_0^4 - f^2\omega_0^4)$$

substitution: $\gamma = \omega^2$

$$\Rightarrow \gamma_{1/2} = (1+f)\omega_0^2 \pm \sqrt{(1+f)^2\omega_0^4 - ((1+f)^2\omega_0^4 - f^2\omega_0^4)}$$

$$= (1+f)\omega_0^2 \pm f\omega_0^2$$

$$\Rightarrow \gamma_1 = \omega_0^2$$

$$\gamma_2 = \omega_0^2 + 2f\omega_0^2 = (1+2f)\omega_0^2$$

resubstitution:

$$\omega_1 = \omega_0 \; ; \quad \omega_2 = -\omega_0$$

$$\omega_3 = \sqrt{1+2f}\,\omega_0 \; ; \quad \omega_4 = -\sqrt{1+2f}\,\omega_0$$

The eigenvectors can be calculated by putting this eigenvalues in equation (1):

$$\omega_1 \text{ and } \omega_2 \Rightarrow z_{1/2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_3 \text{ and } \omega_4 \Rightarrow z_{3/4} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Analysis

Assignment 1:

The measured frequencies are:

left pendulum: $(0{,}52 \pm 0{,}01)\ Hz$
right pendulum: $(0{,}52 \pm 0{,}01)\ Hz$

As already mentioned, this values can be called identically.

The theoretical value can be calculated as follows:

$$s = \left[\frac{l_1 + l_2}{2} m_0 + \left(l_1 + x + \frac{h_1}{2}\right) m_1 + m_2 \left(l_1 + x + h_1 + \ldots \right. \right.$$

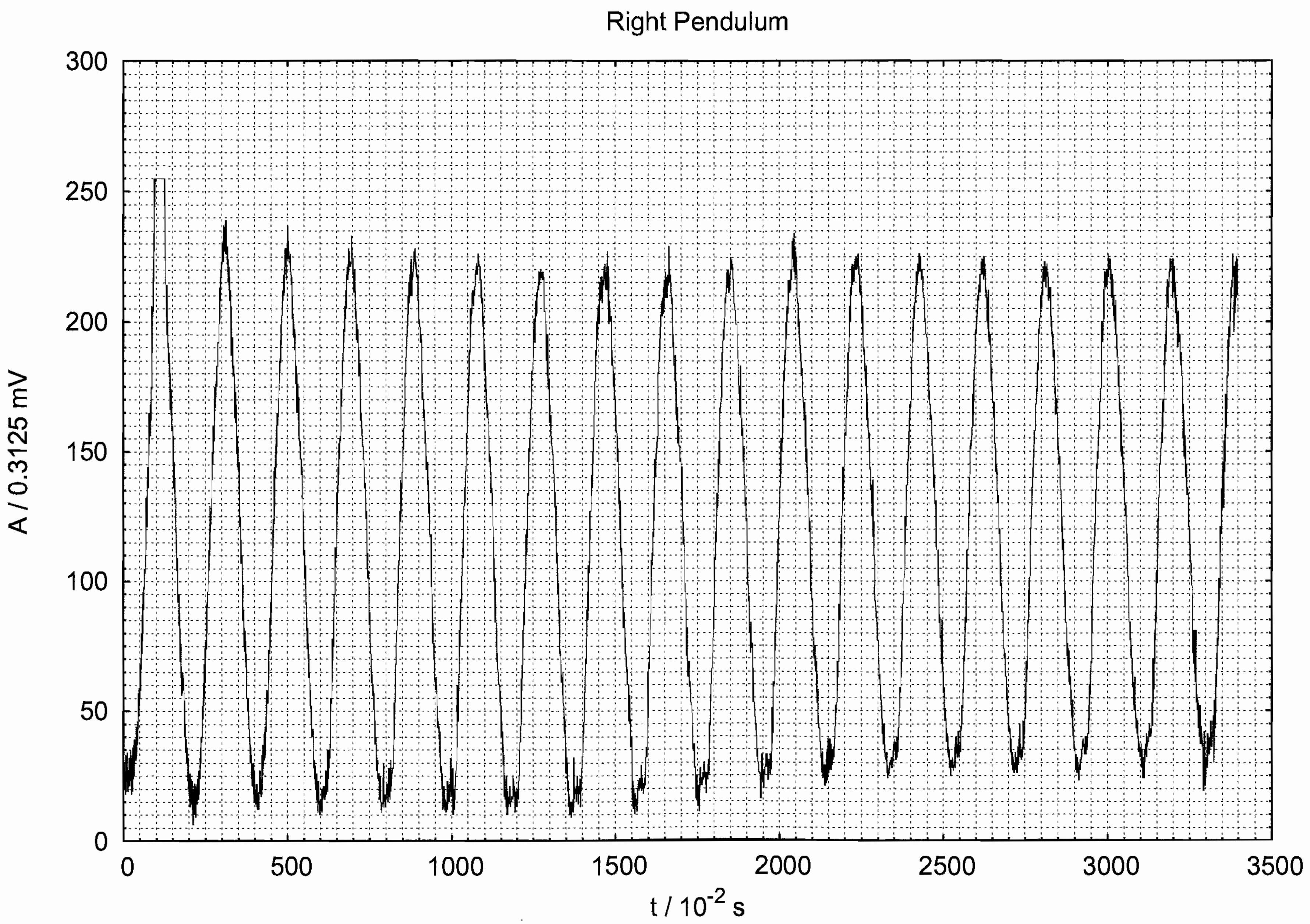
$$\cdot \frac{1}{m_0 + m_1 + m_2} - l_1$$

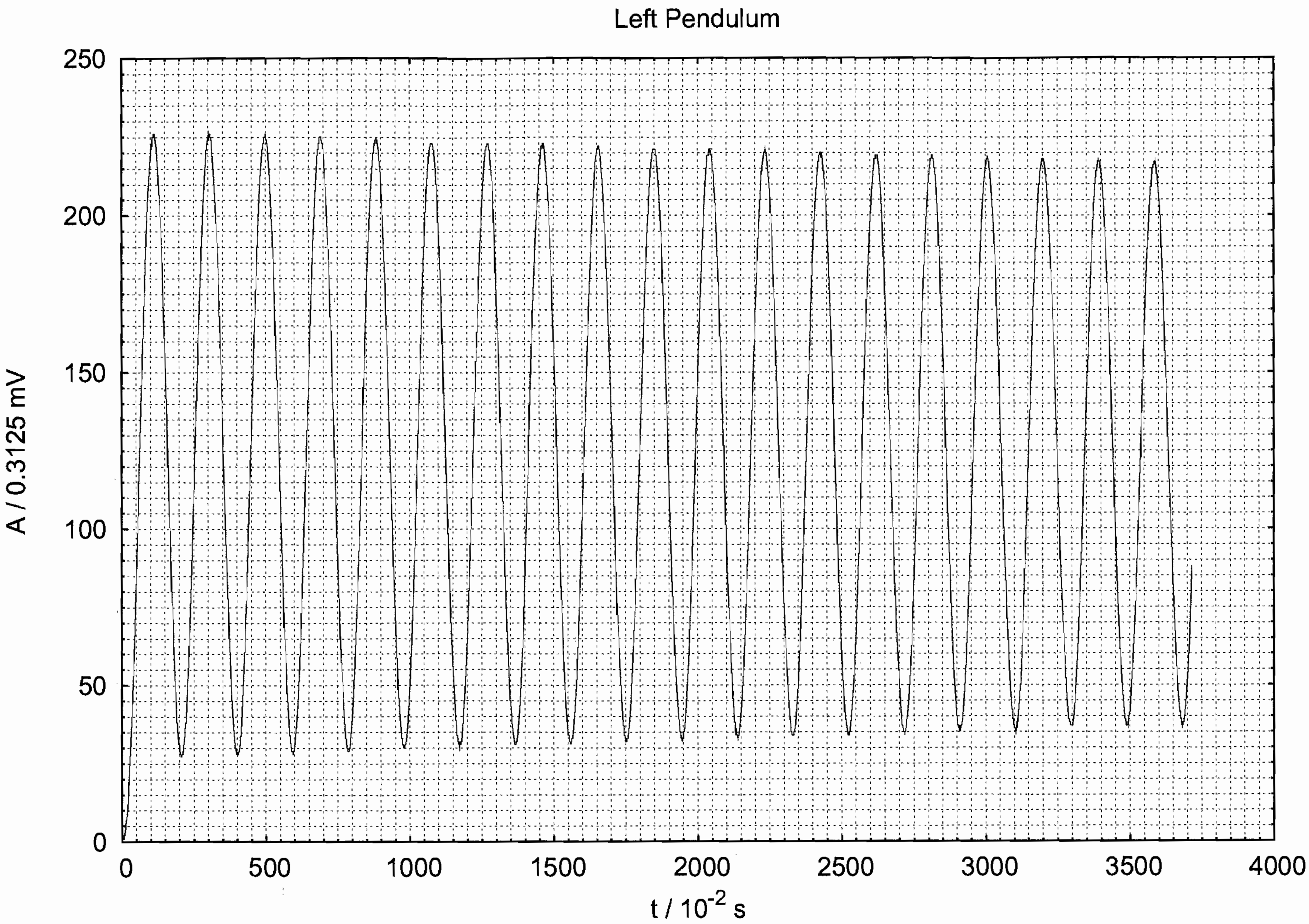
$$\Rightarrow s = (0{,}813 \pm 0{,}005)\ m$$

From this the theoretical value of ω_0 can be calculated to be

$$\nu_0 = (0{,}55 \pm 0{,}01)\ Hz$$

This values can be called compatible within error.


Right Pendulum
A / 0.3125 mV
300
250
200
150
100
50
0
0
500
1000
1500
2000
2500
3000
3500
t / 10^-2 s


Left Pendulum
A / 0.3125 mV
0
50
100
150
200
250
0
500
1000
1500
2000
2500
3000
3500
4000
t / 10^-2 s

Assignment 2:

From the following plots we get the frequency of the symmetric and asymmetric mode for each couplingpoint.

The coupling degree can be calculated as follows:

$$\omega_a = \sqrt{1 + 2f}\,\omega_s$$

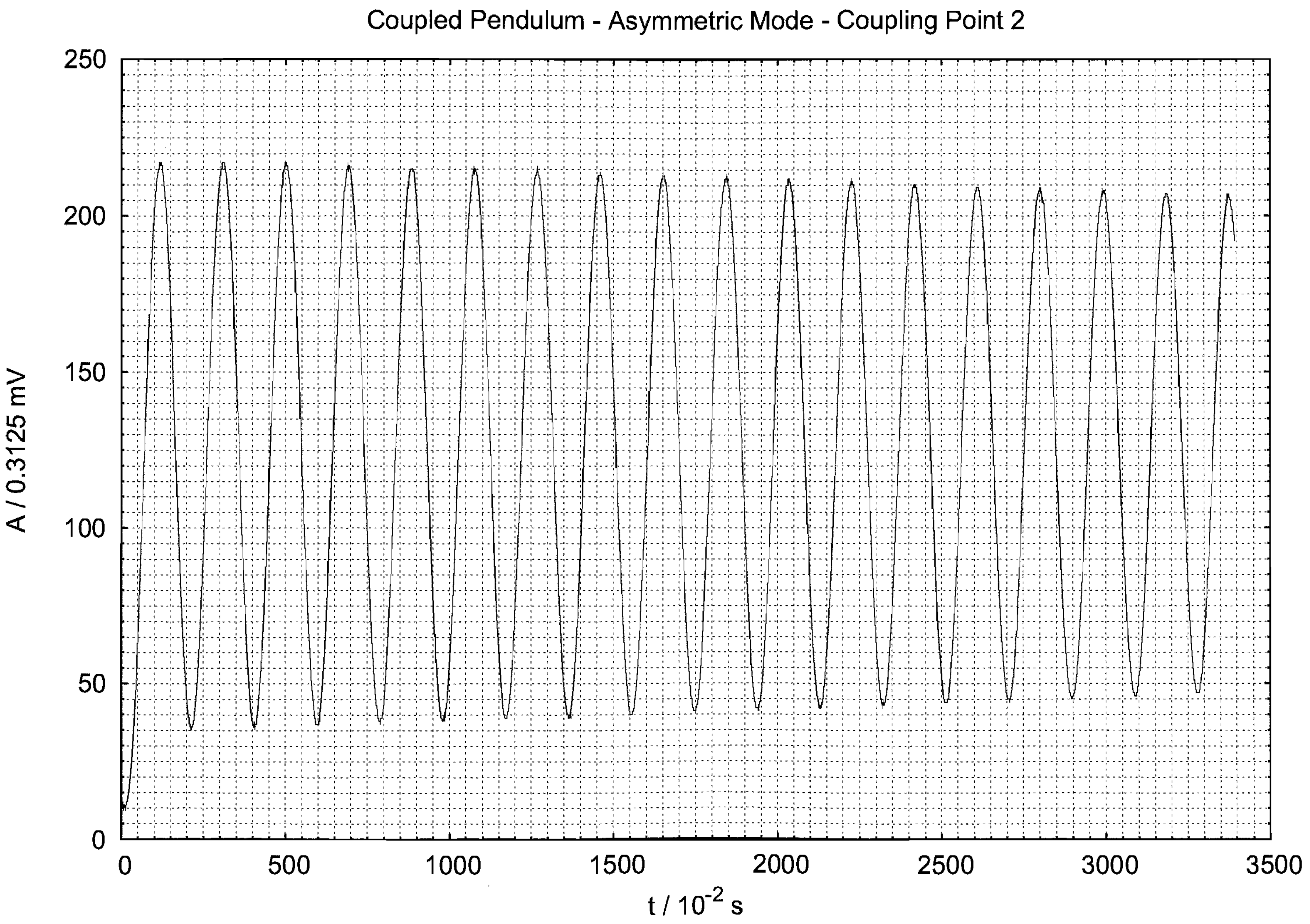
$$\Leftrightarrow \omega_a^2 = (1 + 2f)\,\omega_s^2$$

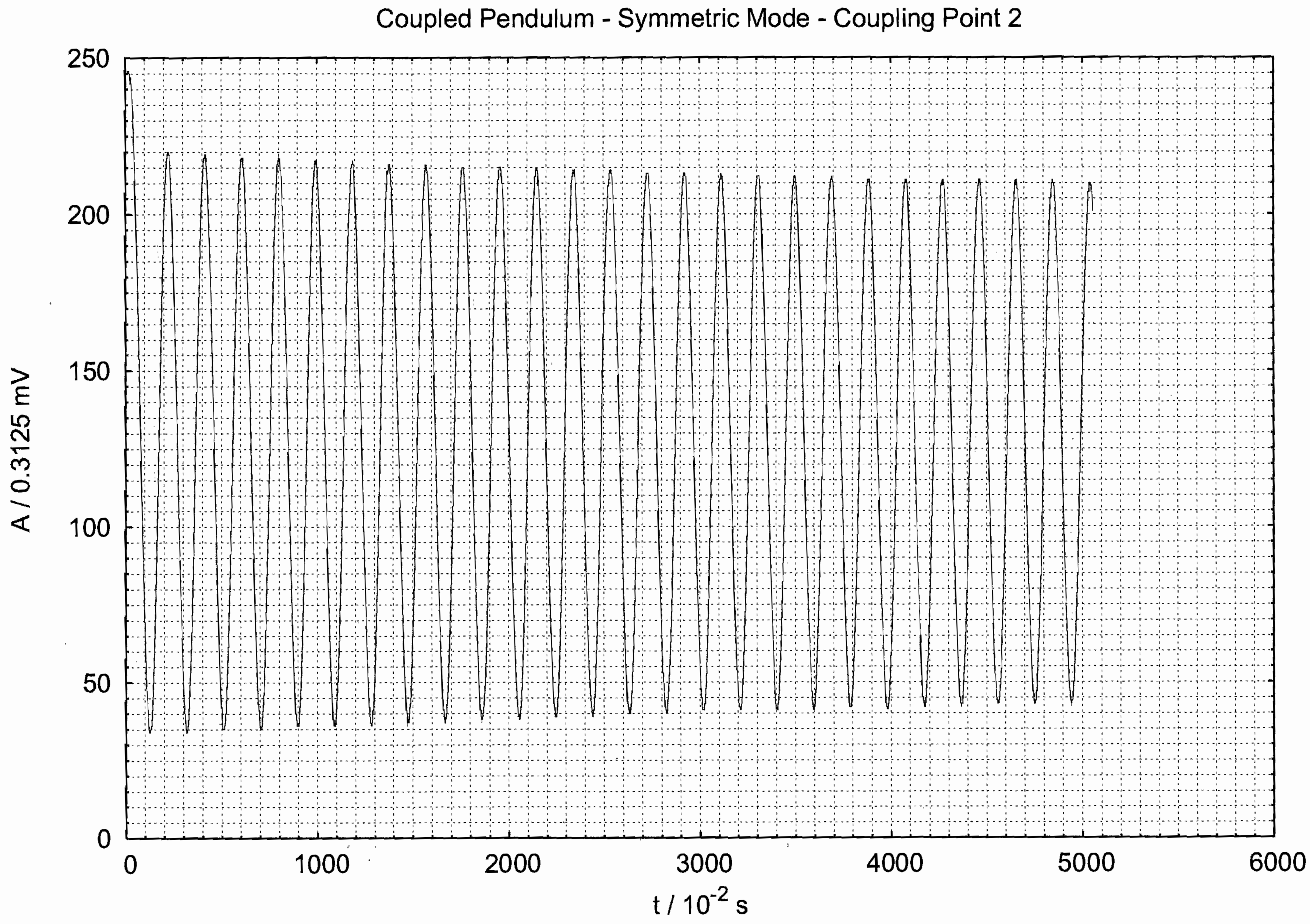
$$\Rightarrow 2f = \left(\frac{\omega_a}{\omega_s}\right)^2 - 1$$

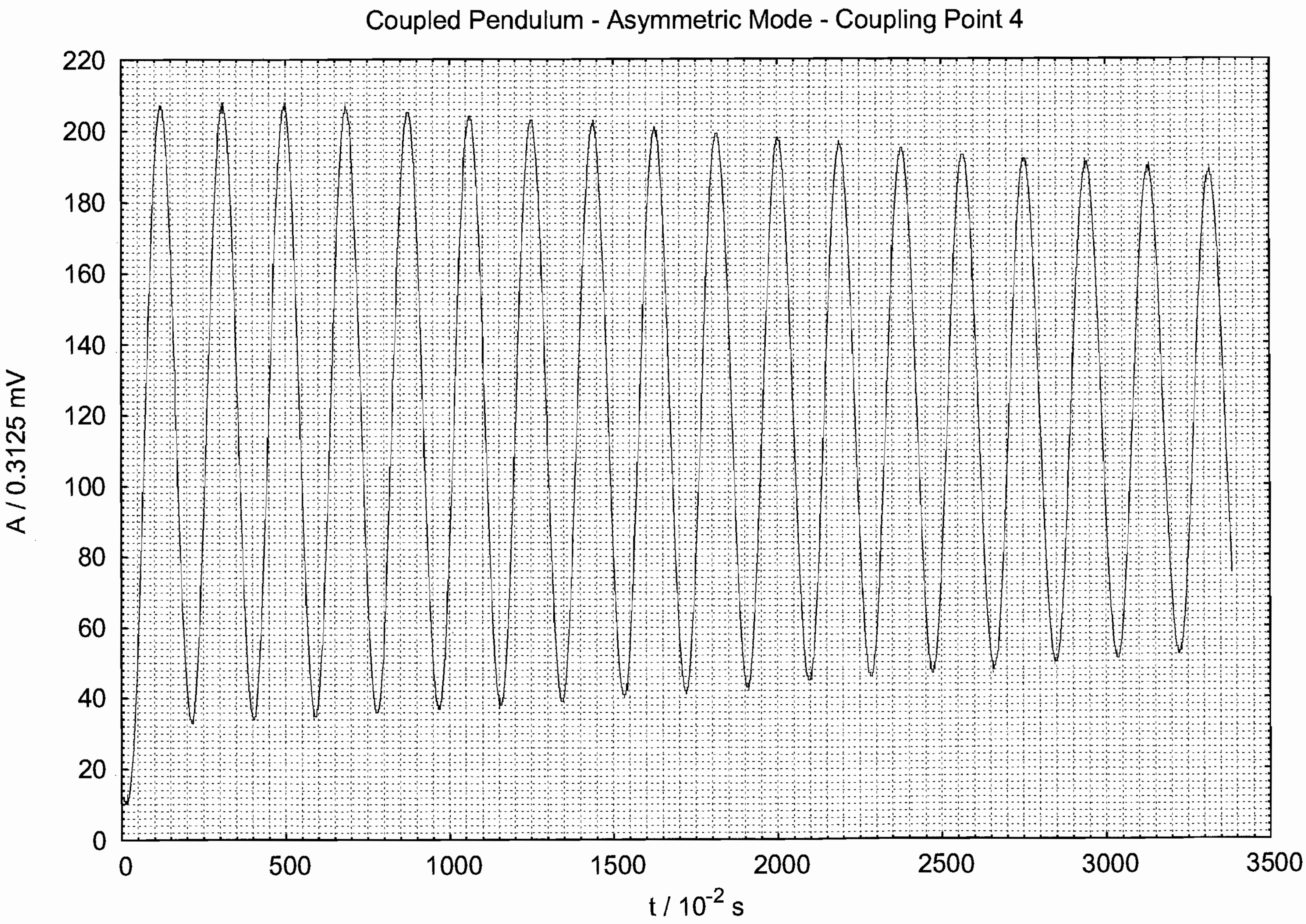
$$\Rightarrow f = \frac{1}{2}\left(\left(\frac{\omega_a}{\omega_s}\right)^2 - 1\right)$$

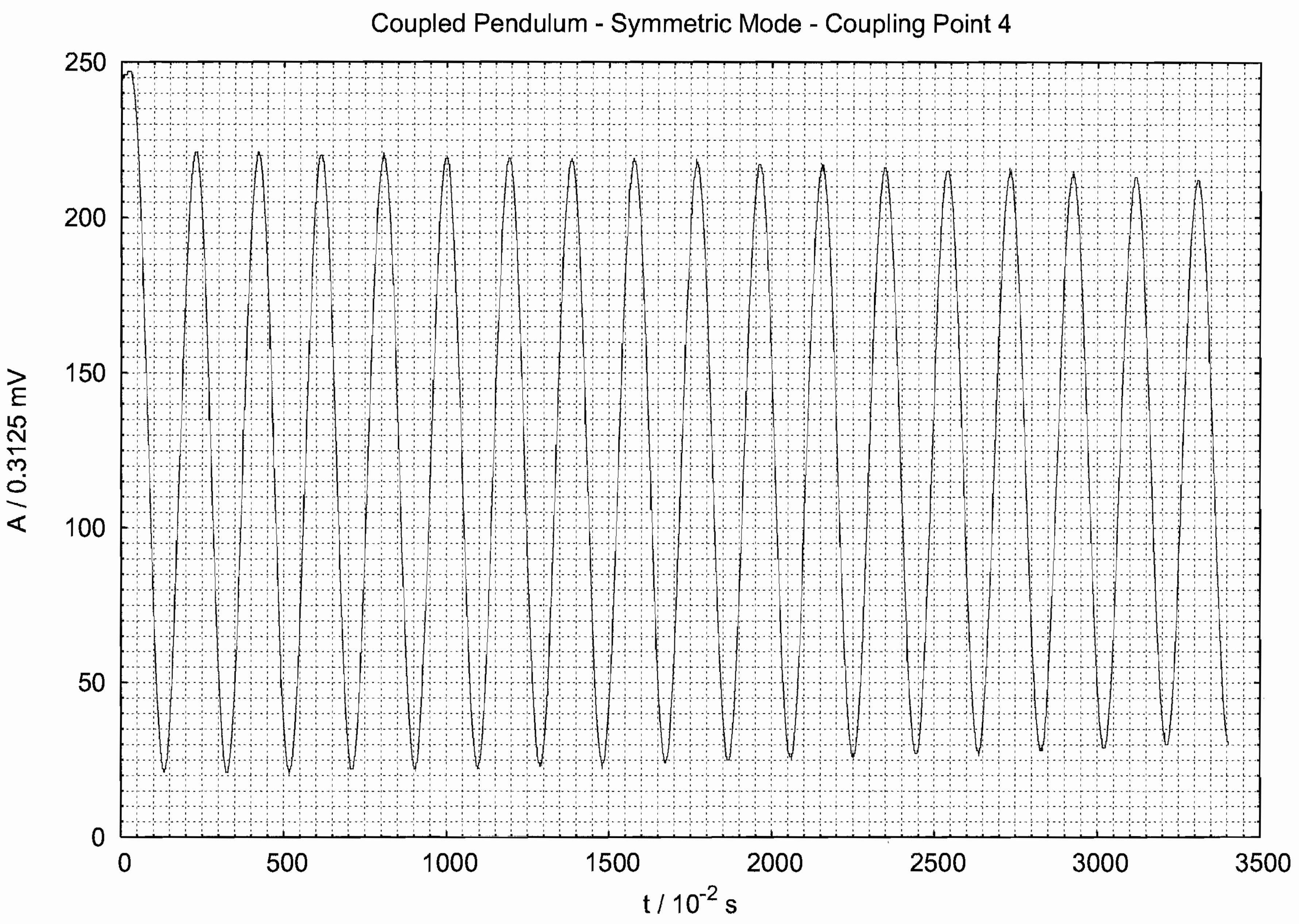
The results are presented in the following table:

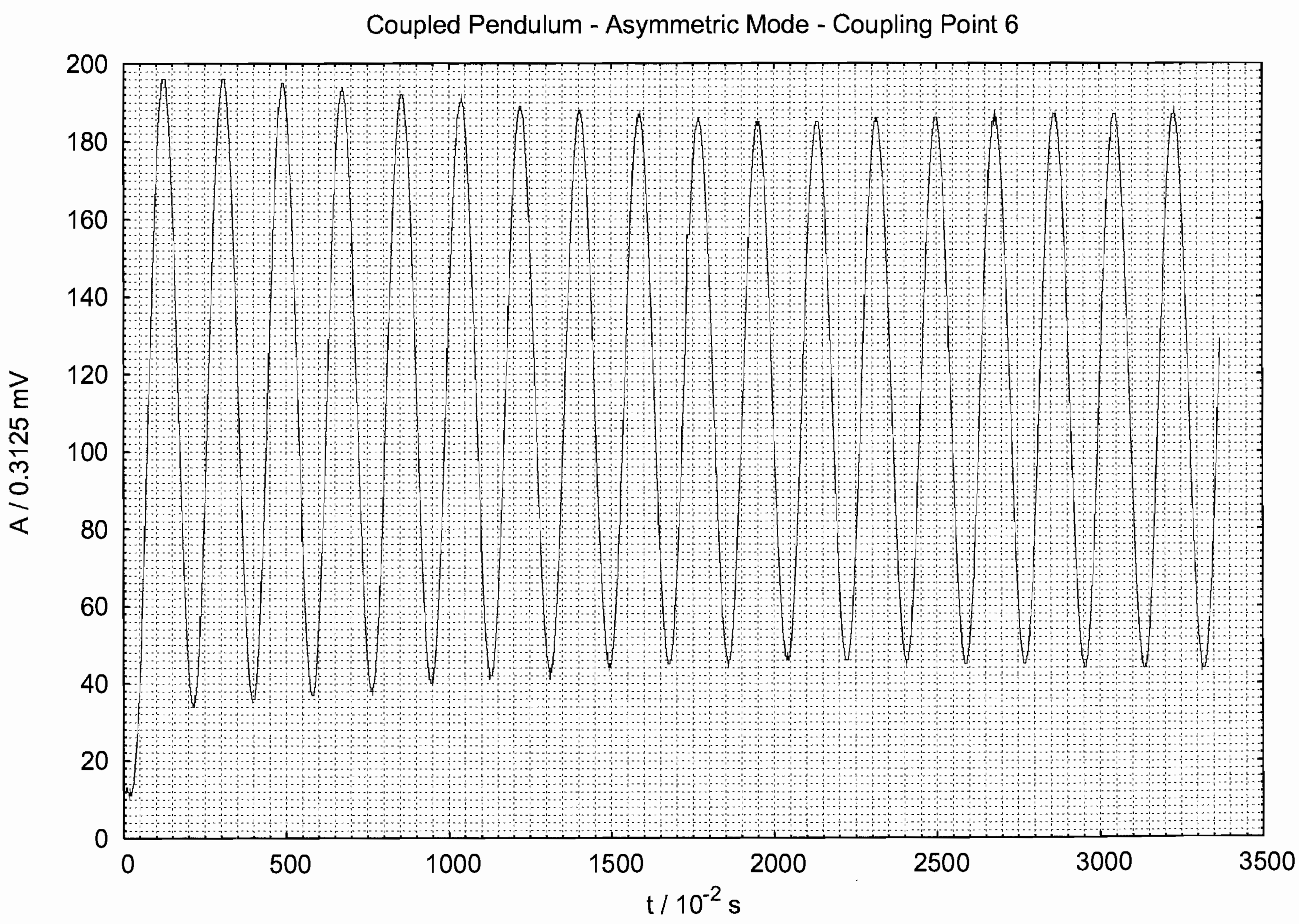
cpl. point [mm]	freq. (s) [Hz]	err [Hz]	freq. (a) [Hz]	err [Hz]	ω (s) [rad/s]	err [rad/s]	ω (a) [rad/s]	err [rad/s]	f	err
200.0 ± 0.2	0.519	0.016	0.522	0.016	3.261	0.098	3.280	0.098	0.006	0.001
400.0 ± 0.4	0.519	0.016	0.533	0.016	3.261	0.098	3.349	0.100	0.027	0.002
600.0 ± 0.6	0.519	0.016	0.545	0.016	3.261	0.098	3.424	0.103	0.051	0.003
700.0 ± 0.7	0.519	0.016	0.558	0.017	3.261	0.098	3.506	0.105	0.078	0.004
800.0 ± 0.8	0.522	0.016	0.567	0.017	3.280	0.098	3.563	0.107	0.090	0.005

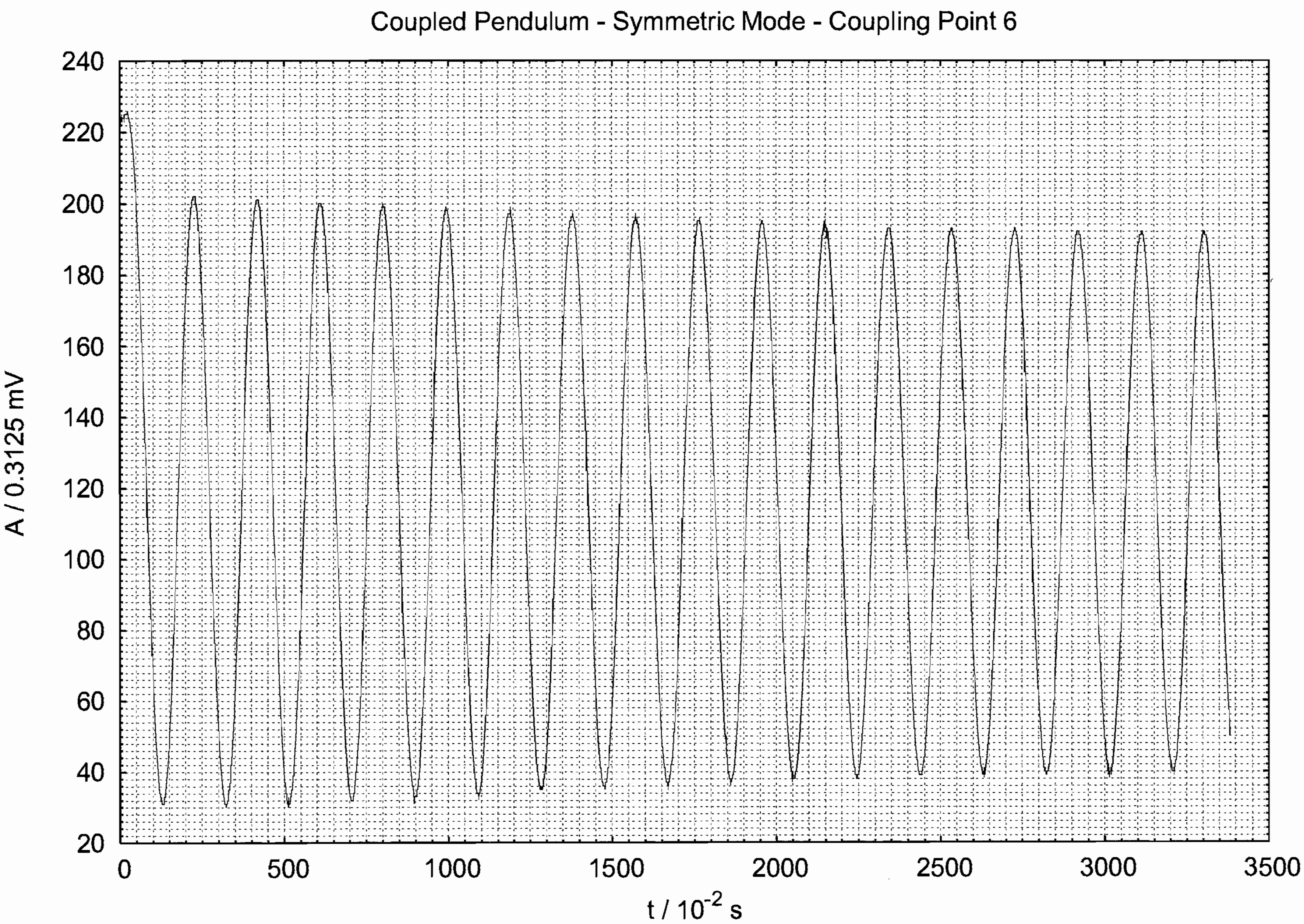

Coupled Pendulum - Asymmetric Mode - Coupling Point 2
A / 0.3125 mV
0
50
100
150
200
250
0
500
1000
1500
2000
2500
3000
3500
t / 10-2 s

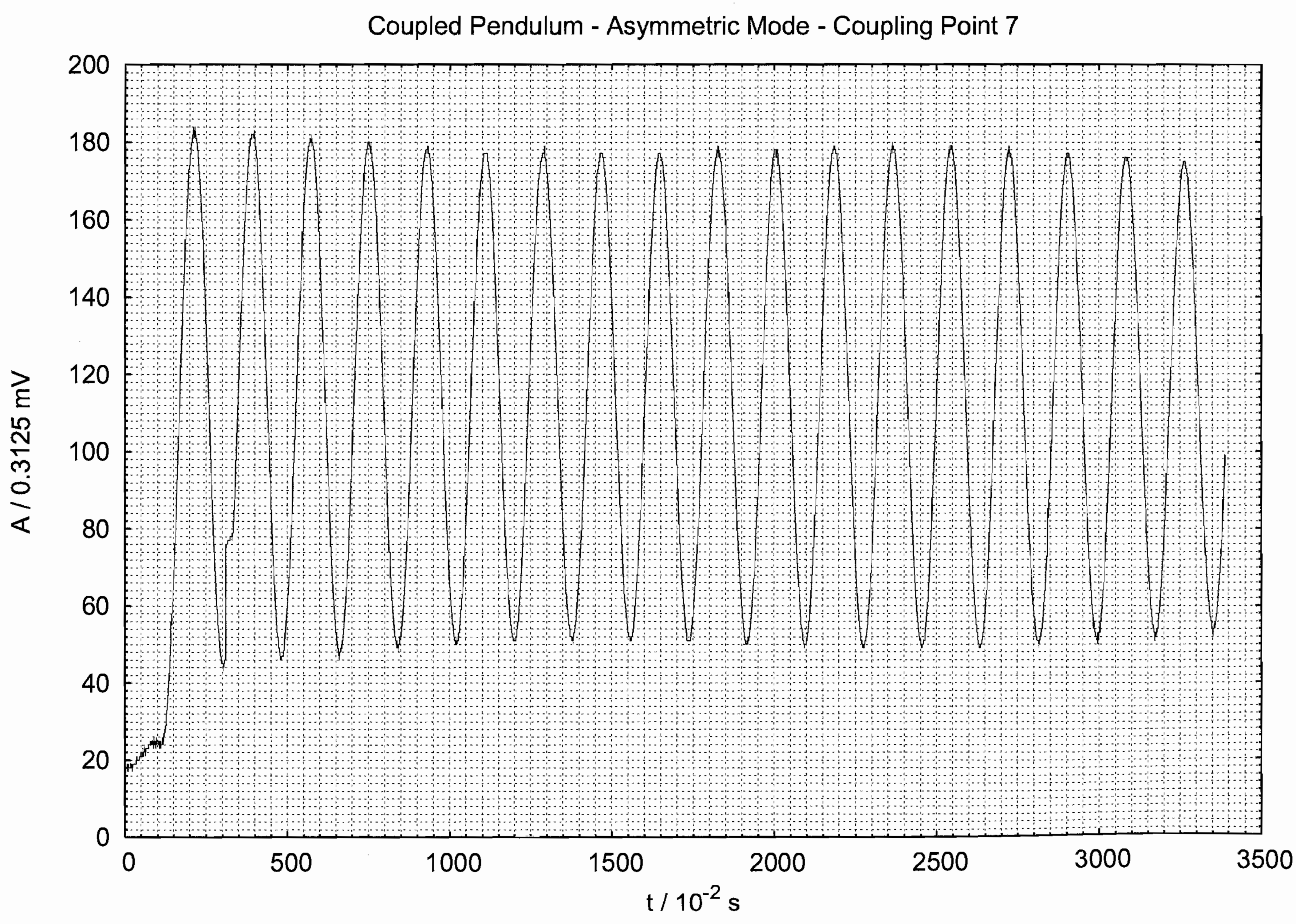

Coupled Pendulum - Symmetric Mode - Coupling Point 2
A / 0.3125 mV
250
200
150
100
50
0
0
1000
2000
3000
4000
5000
6000
t / 10-2 s


Coupled Pendulum - Asymmetric Mode - Coupling Point 4
A / 0.3125 mV
t / 10-2 s
0
20
40
60
80
100
120
140
160
180
200
220
0
500
1000
1500
2000
2500
3000
3500

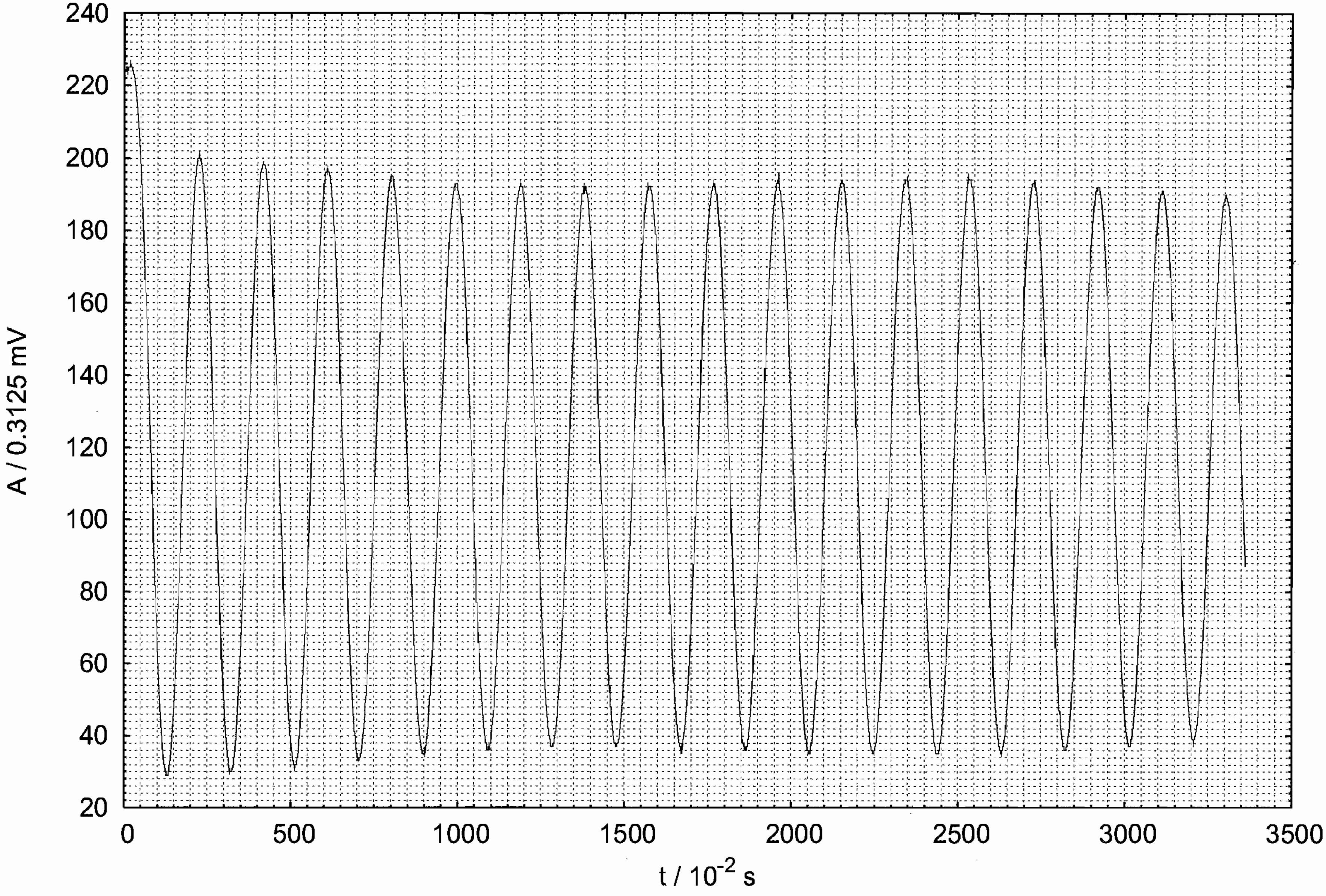

Coupled Pendulum - Symmetric Mode - Coupling Point 4
A / 0.3125 mV
250
200
150
100
50
0
500
1000
1500
2000
2500
3000
3500
t / 10^-2 s

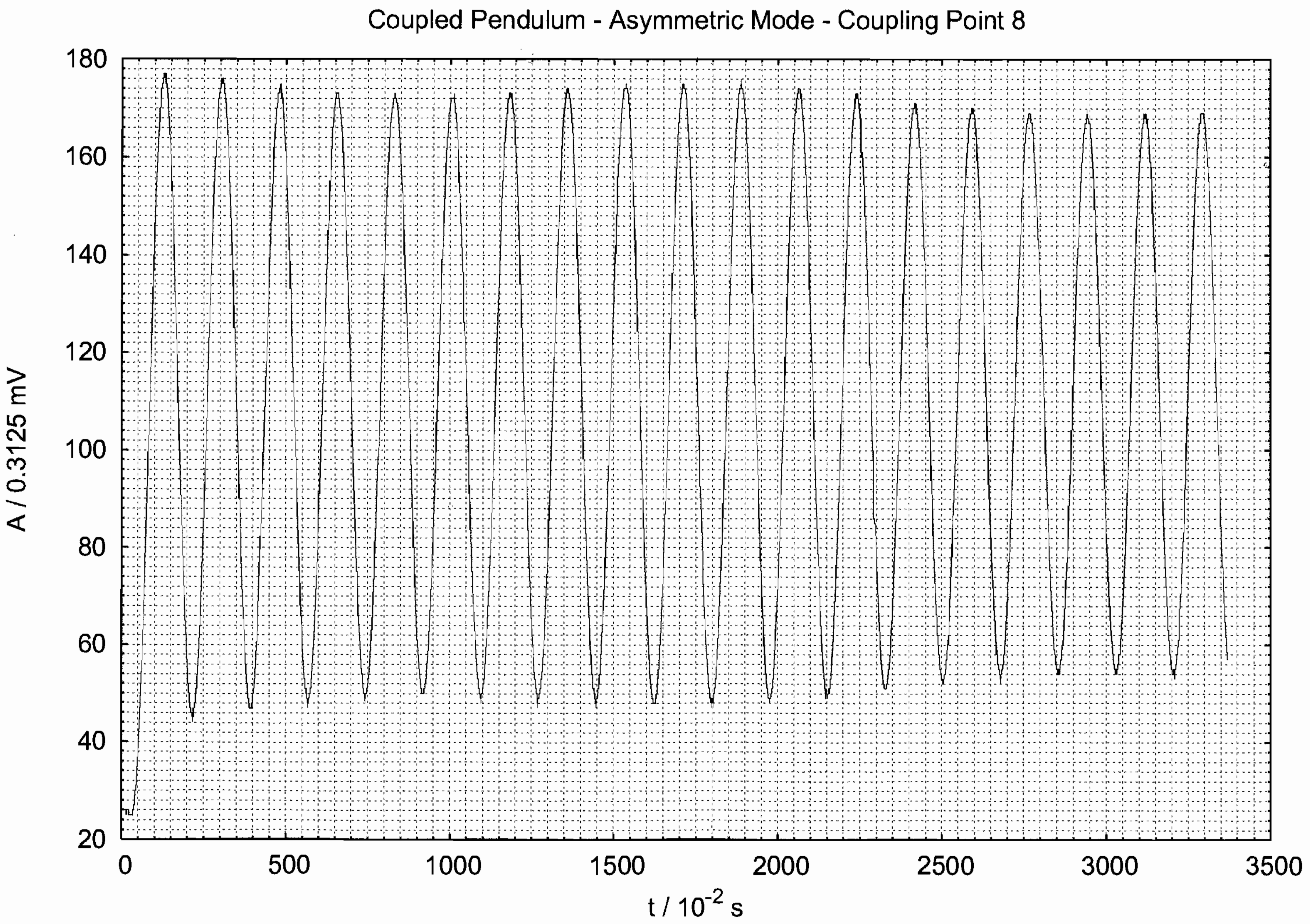

Coupled Pendulum - Asymmetric Mode - Coupling Point 6
A / 0.3125 mV
t / 10^-2 s
0
20
40
60
80
100
120
140
160
180
200
0
500
1000
1500
2000
2500
3000
3500


Coupled Pendulum - Symmetric Mode - Coupling Point 6
A / 0.3125 mV
t / 10^-2 s
240
220
200
180
160
140
120
100
80
60
40
20
0
500
1000
1500
2000
2500
3000
3500

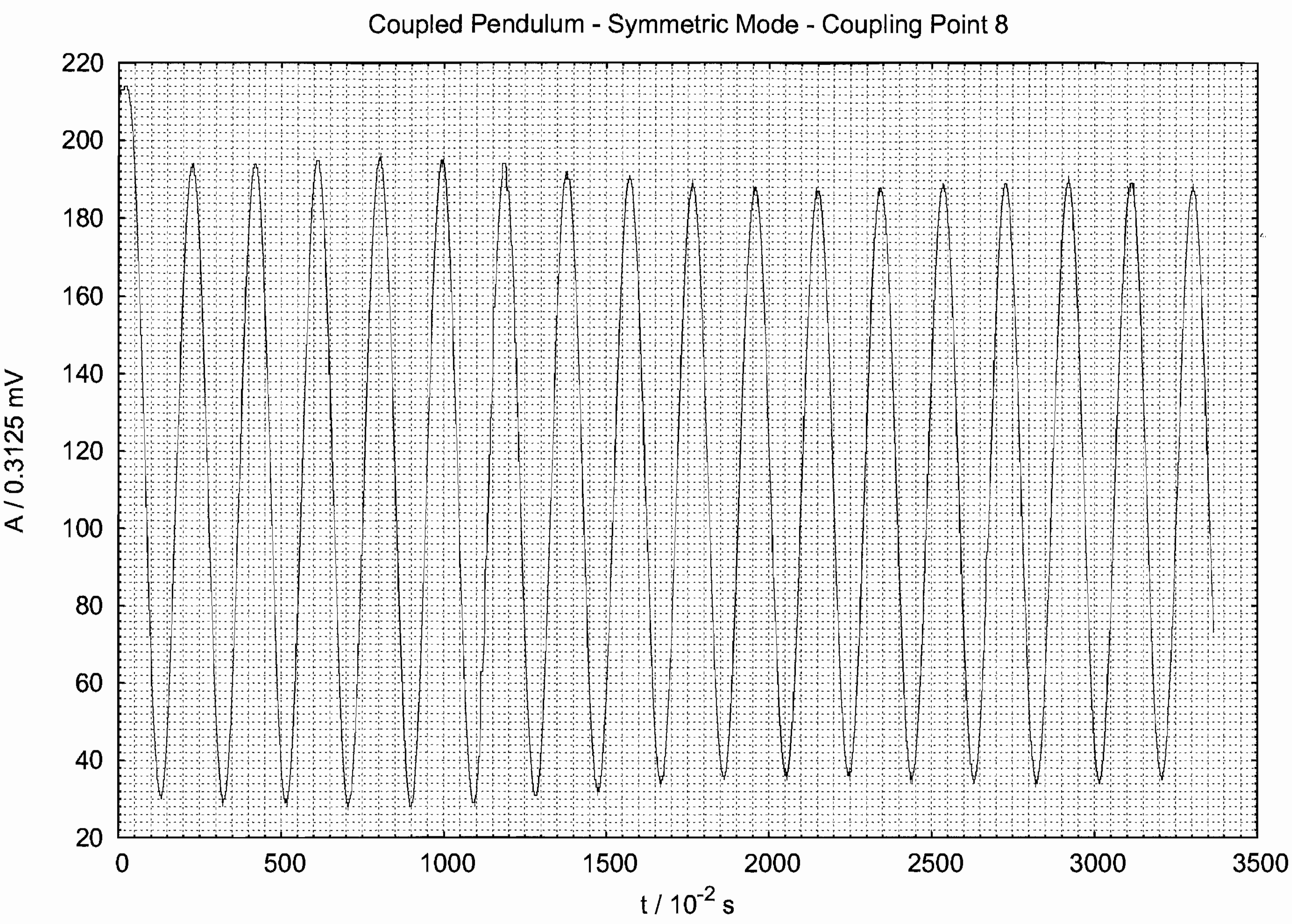

Coupled Pendulum - Asymmetric Mode - Coupling Point 7
A / 0.3125 mV
0
20
40
60
80
100
120
140
160
180
200
0
500
1000
1500
2000
2500
3000
3500
t / 10^-2 s

Coupled Pendulum - Symmetric Mode - Coupling Point 7





Coupled Pendulum - Asymmetric Mode - Coupling Point 8
A / 0.3125 mV
t / 10^-2 s
180
160
140
120
100
80
60
40
20
0
500
1000
1500
2000
2500
3000
3500



Coupled Pendulum - Symmetric Mode - Coupling Point 8
A / 0.3125 mV
220
200
180
160
140
120
100
80
60
40
20
0
500
1000
1500
2000
2500
3000
3500
t / 10-2 s

Assignment 3:

The beat frequency can be read from the following plots.

From this the degrees of coupling can be calculated ~~usig~~ using the following equations:

$$(1) \quad \Delta\omega = \frac{1}{2}(\omega_a - \omega_s)$$

$$\Leftrightarrow 2\Delta\omega = \omega_a - \omega_s$$

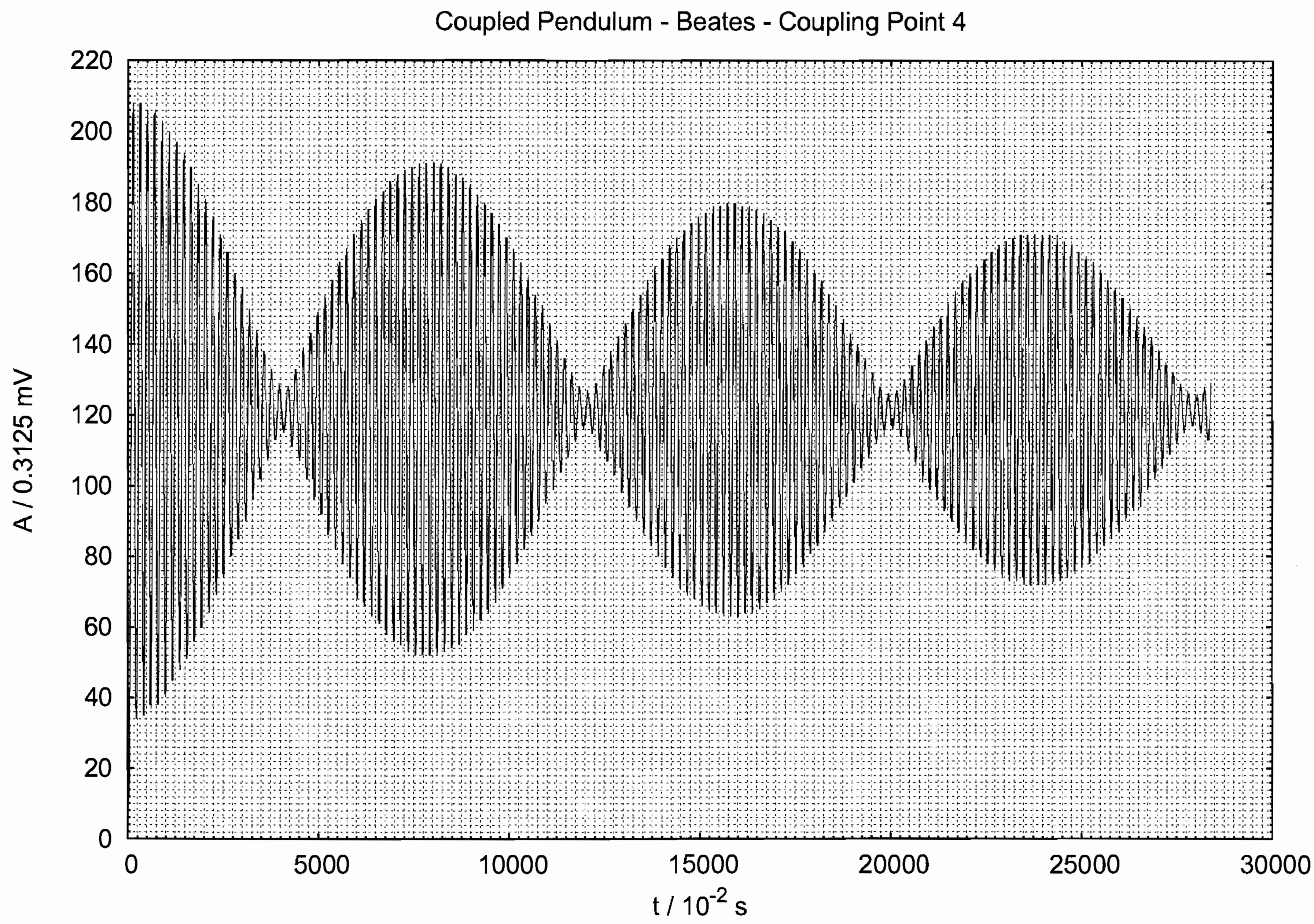
$$\Leftrightarrow \omega_a = 2\Delta\omega + \omega_s$$

$$\Leftrightarrow \omega_a = 2\Delta\omega + \omega_0$$

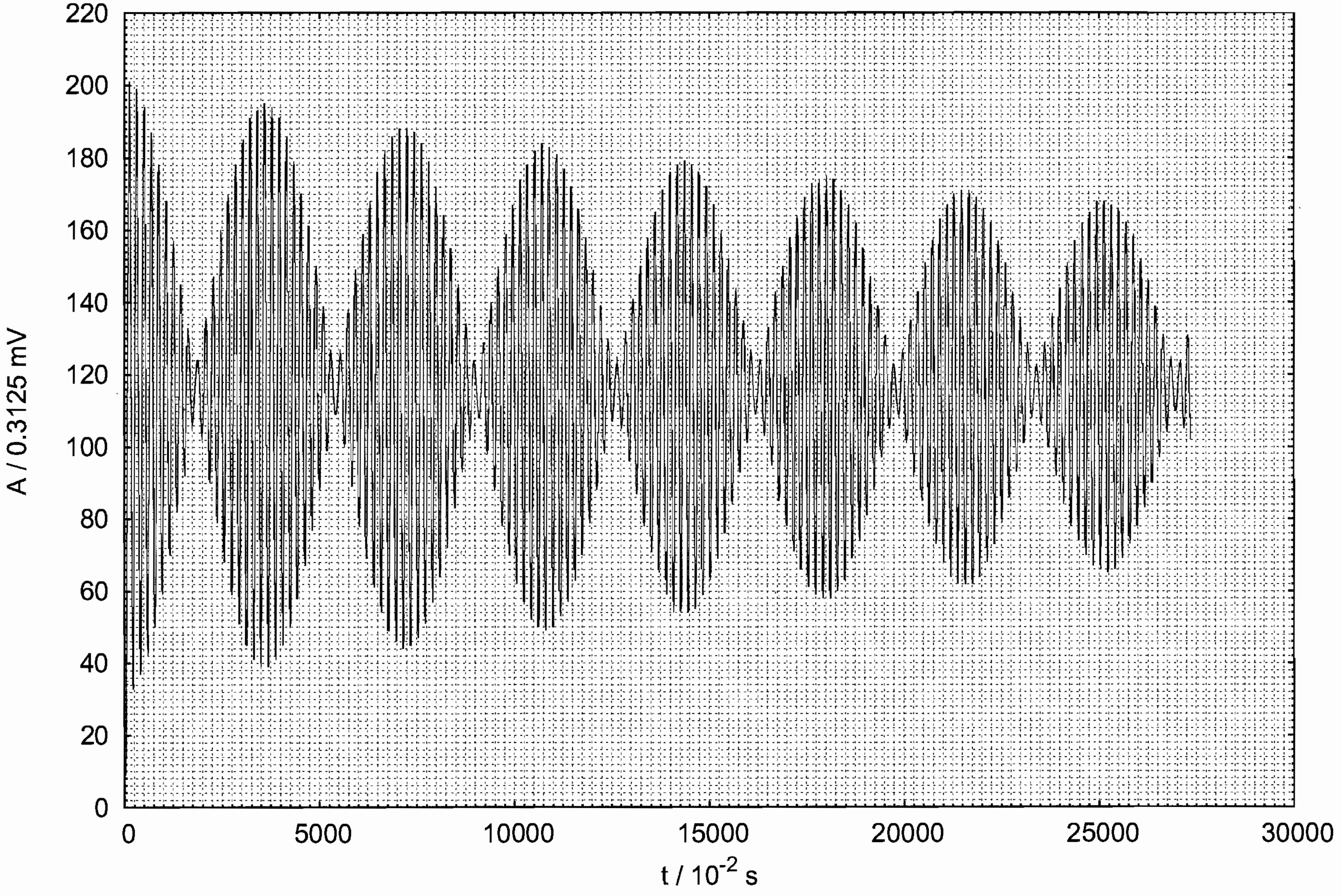
$$(2) \quad f = \frac{1}{2}\left(\left(\frac{2\Delta\omega + \omega_0}{\omega_0}\right)^2 - 1\right)$$

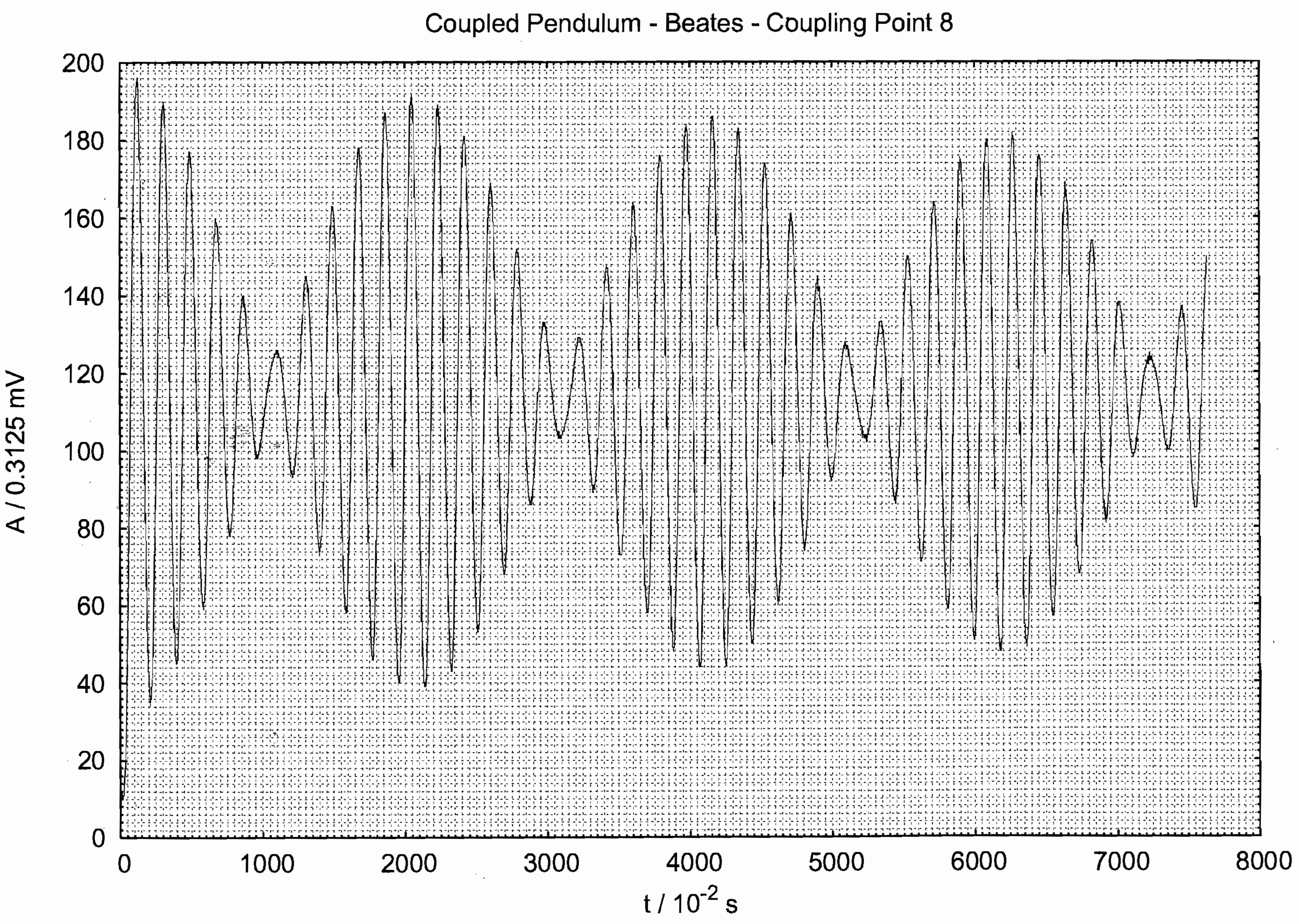
This formula leads to the degrees of coupling, calculated in the table below:

cpl. point [mm]	basic freq. [Hz]	err [Hz]	beat freq. [Hz]	err [Hz]	ω (basic) [rad/s]	err [rad/s]	ω (beats) [rad/s]	err [rad/s]	f	err
400.0 ± 0.4	0.520	0.010	0.00625	0.00013	3.267	0.065	0.039	0.001	0.024	0.001
600.0 ± 0.6	0.520	0.010	0.01380	0.00028	3.267	0.065	0.087	0.002	0.054	0.002
800.0 ± 0.8	0.520	0.010	0.02470	0.00049	3.267	0.065	0.155	0.003	0.100	0.003


Coupled Pendulum - Beates - Coupling Point 4
A / 0.3125 mV
t / 10^-2 s
0
20
40
60
80
100
120
140
160
180
200
220
0
5000
10000
15000
20000
25000
30000

Coupled Pendulum - Beates - Coupling Point 6




Coupled Pendulum - Beates - Coupling Point 8
A / 0.3125 mV
200
180
160
140
120
100
80
60
40
20
0
1000
2000
3000
4000
5000
6000
7000
8000
t / 10^-2 s

Assignment 4:

The plot on the next page shows a fit and an error fit of the linear behavior of the degrees of coupling over r^2. From the theory we get:

$$\frac{f}{r^2} = \frac{D}{m \cdot g \cdot s}$$

$$\Leftrightarrow D = m \cdot g \cdot s \cdot \left(\frac{f}{r^2}\right)$$

Therefore we get the spring constant from the product of the slope, the mass, gravity constant and the distance from point of rotation to the center of mass (s).

$$D = (2{,}91 \pm 0{,}30)\ \frac{kg}{s^2}$$

~~The~~ Another method is to measure the spring constant directly with the spring oscillation:

$$\omega = \sqrt{\frac{D}{m}}$$

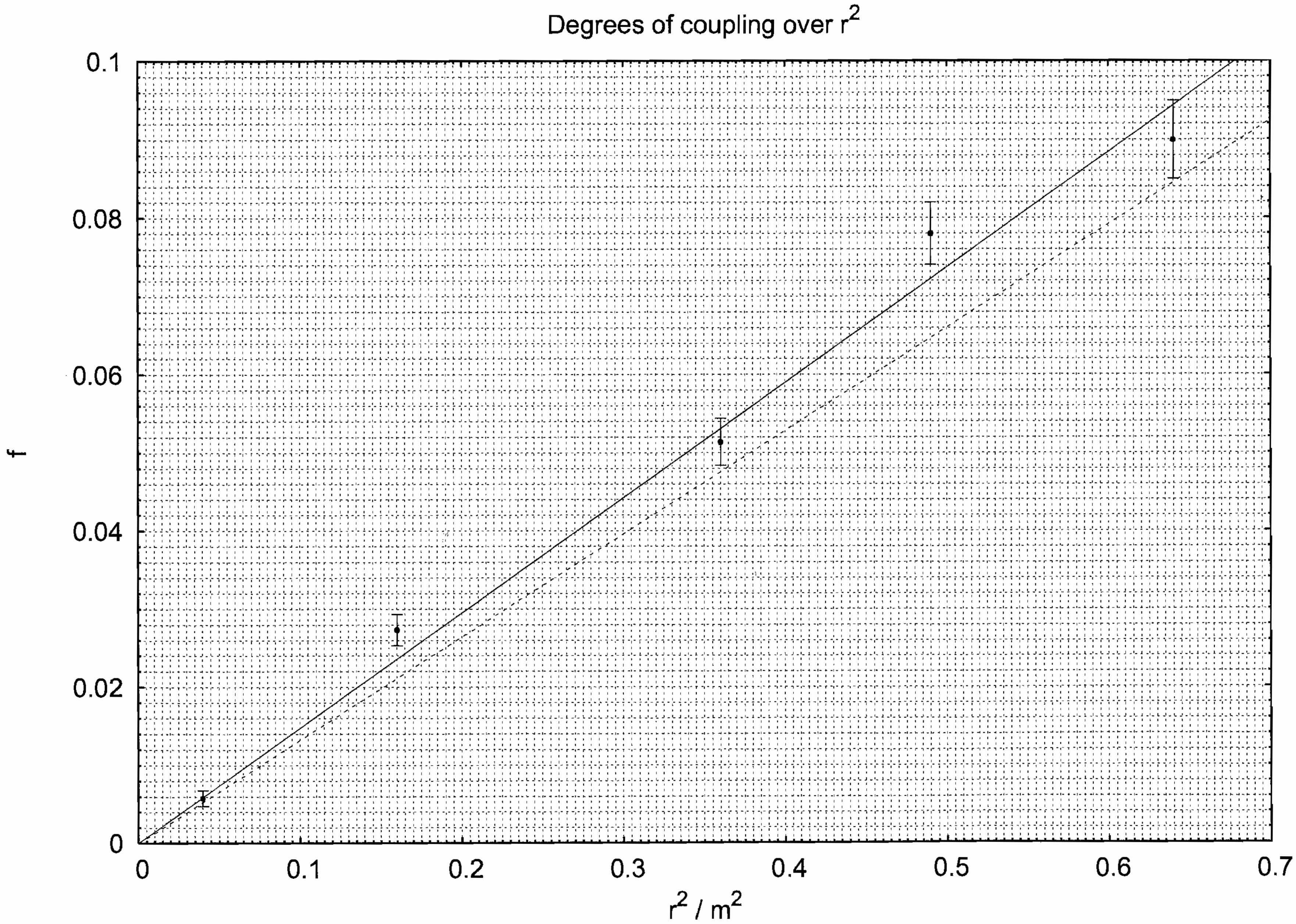
$$\Leftrightarrow D = \omega^2 \cdot m$$

The frequency ω was measured to be:

$$\omega = (7{,}29 \pm 0{,}29)\ \frac{rad}{s}$$

From this follows: $D = (2{,}66 \pm 0{,}21)\ \frac{kg}{s^2}$

The values are compatible within error.


Degrees of coupling over r^2
f
0.1
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
r^2 / m^2

Conclusion

The propose of this experiment was to examine the general behavior of coupled oscillations. First of all, two pendulums with the same frequency, mass and length where necessary. In the first measurement we compared the frequencies of our pendulums. These have been identical, which was an important initial condition. However there was a small difference between the measured values and the calculated one, which might have been caused by friction for example. The second measurement was to compare the symmetric an asymmetric oscillation mode. As we expected in the introduction, there was no influence of the spring during the symmetric oscillation mode. The frequency was identically to the one we measured in assignment one. ~~In~~ As a result we get first values ~~for~~ the degrees of coupling. The method used during assignment 3 leaded to additional values for the same coupling points. These ~~were~~ are compatible the the one before. In both cases the error was determined by the reading accuracy. The last part of the experiment results into a value for the spring constant. Again we used two different methods to measure that quantity. The high error of about 8 - 10 % can be explained by the barely linear behaviour of the degrees of coupling on the one hand and the mass of the spring which haven't been considered on the other hand.

⇒ Missing discussion of plots.

(4,5)

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