Harmonic Oscillations

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Introduction

Harmonic oscillators are usually described in three variations. In the simplest case, there is no external force and no friction. Secondly, there can be still no external force but a friction that is proportional to velocity. In the third and most complicated case, there is both an external (harmonic) force and friction.

The basic harmonic oszillator only involves a repelling force that is directly proportional to displacement (but in opposite direction)

$F = m\ddot{x} = -Dx$ which is easily solved with the ansatz $x(t) = e^{i\omega t}$ as $x(t) = A_0 \cos(\omega_0 t) + A_1 \sin(\omega_0 t)$ with the angular frequency

$$\omega_0 = \sqrt{\frac{D}{m}}$$

When we take friction into account, we get one more term in the differential equation:

$$m\ddot{x} + k\dot{x} + Dx = 0$$

Again, we can use the same ansatz $x(t) = x^{i\omega t}$

$$\Rightarrow \; -\omega^2 m x + i\omega k x + Dx = 0$$

$$\Leftrightarrow \; \left(\omega - \frac{ik}{2m}\right)^2 = \frac{D}{m} - \frac{k^2}{4m^2}$$

$$\Rightarrow \; \omega_{1,2} = i\frac{k}{2m} \pm \sqrt{\frac{D}{m} - \frac{k^2}{4m^2}}$$

~~ω is again the angular frequency of the resulting oscillation~~

We can structure this result as following:

$$\omega_{1/2} = i \underbrace{\frac{k}{2m}}_{\delta} \pm \underbrace{\sqrt{\underbrace{\frac{D}{m}}_{\omega_0^2} - \underbrace{\frac{k^2}{4m^2}}_{\delta^2}}}_{\omega_1} = i\delta \pm \omega_1$$

When we insert this back into the ansatz, we will find that there is a harmonic oscillation only if ω_1 has no imaginary part, i.e. $\omega_0^2 > \frac{k^2}{4m^2}$. In any other case, the oscillator will return to its position of rest only. If the friction is small enough, however, and allows oscillation, the movement can be described as

$$x(t) = e^{-\delta t}\left[A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right] \quad \text{with}$$

$A_1 = x_0$ and $A_2 = \frac{\delta x_0 + \dot{x}}{\omega_0}$ or, more conveniently

$$x(t) = A \cdot e^{-\delta t} \cos(\omega_1 t + \beta)$$

When we add an external driving force, we expect the system to adapt the driving frequency (forced oscillation). Mathematically, we have to solve an inhomogeneous differential equation

$$m\ddot{x} + k\dot{x} + Dx = F_0 \cos(\Omega t)$$

To solve it, we must find one particular solution, which we then add to the homogeneous solution.
As an ansatz, we choose $x(t) = A_s \cos(\Omega t + \phi)$,

or, in ~~complex~~ imaginary form:

$$\cancel{\Xi} \quad x(t) = A_s \cdot e^{i(\Omega t + \phi)}$$

When we insert this ansatz, we find

$$A_s = \frac{F_0/m}{\sqrt{(\omega_0^2-\Omega)^2+4\delta^2\Omega}}$$

$$\phi = \text{Arctan}\left(\frac{-2\delta\Omega}{\omega_0^2-\Omega^2}\right)$$

After a sufficiently long time, this is the only remaining part of the solution, as the homogeneous part decreases exponentially

During the initial phase, when this term still has a visible effect, we have to consider the full solution

$$x(t) = A \cdot e^{-\delta t}\cos(\omega_0 t + \beta) + A_s(\Omega)\cos(\Omega t + \phi)$$

The resulting oscillation can be very complicated.

With certain initial conditions we can find a simpler solution

If $\delta \ll \omega_0$, $\Omega \simeq \omega_0$, $x_0 = 0$, $\dot{x}_0 = 0$

$$\Rightarrow x(t) = A_s[\cos(\Omega t + \phi) - e^{-\delta t}\cos(\omega_0 t + \beta)]$$

This results in beats with a frequency of $\frac{|\Omega - \omega_0|}{2}$

If the two cosine terms ~~have~~ are equal with opposite sign, the maximum initial amplitude is reached

If $\Omega = \omega_0$, we find that

$$x(t) = A_s(1 - e^{-\delta t}) \sin \omega_0 t$$

As we have seen A_s is dependant on Ω. It has a peak close to ω_0, the width of the curve is determined by the friction coefficient

→

If this coefficient is small, the amplitude can reach very high values at the resonance frequency, and will jump quickly to its peak

If $\omega_0 \approx \Omega$ and $|\omega_0 - \Omega| = \Delta\Omega$ we can write A_S in a simplified way as

$$A_S(\Delta\Omega) = \frac{F_0/m}{2\delta\omega_0\sqrt{1+\left(\frac{\Delta\Omega}{\delta}\right)^2}}$$

The amplitude reaches its maximum at $\Omega = \omega_0$ then and has a value $\frac{A_{max}}{\sqrt{2}}$ at $\Omega = \omega_0 \pm \delta$, which allows to easily measure δ

Assignments

1) Examination of harmonic oscillation with friction, but without external force. Measurement of displacement in dependancy of time. Calculation of the eigen frequency and the system's friction coefficient
2) Examination of forced oscillations. Measurement of displacement in dependancy of frequency. Calculation of eigen frequency and friction coefficient.
3) Qualitative examination of the phase shift between the external force and the oscillator, in dependancy of the external force's frequency
4) Examination of the initial oscillation in the case of resonance and in the vicinity

Experiment

21.03.05

start 9^{45} end 13^{00}

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Assignment 1 : See printout

damping levels 1,2,3

the "undamped" oscillator has a constant friction !

Assignment 2 : ~~see printout~~

measurement at damping level 1

frequency is given as the voltage at the driving motor

the error is from the non-constant amplitude

frequency (V)	amplitude
7	0,0152 ± 0,0020
8	0,0162 ± 0,0020
9	0,0280
10	0,0620
11	0,2800 ± 0,0100
12	0,0520 ± 0,0010
13	0,0320 ± 0,0020
10,2	0,0980 ± 0,0020
10,4	0,1300 ± 0,0100
10,6	0,1700 ± 0,0100
10,8	0,4100
10,7	0,3000
10,9	0,5300
11,1	0,1800
11,2	0,1100

frequency	amplitude
11,3	0,1000 ± 0,0100
11,4	0,0930 ± 0,0030
11,7	0,0610
10,5	0,1500 ± 0,0100
9,5	0,0370 ± 0,0030
11,5	0,085

Conversion Voltage → Hertz

$11{,}5\,V \rightarrow$ 7,8 sec for 5 oscillations

$10{,}9\,V \rightarrow$ 8,5 sec for 5 "

$10{,}6\,V \rightarrow$ 8,7 sec "

$11{,}1\,V \rightarrow$ 8,1 sec

Assignment 3:

at low Ω ($\Omega_V = 7V$) oscillators follows external force

at resonance ($\Omega_V = 10{,}9V$) $\approx 90°$ phase shift

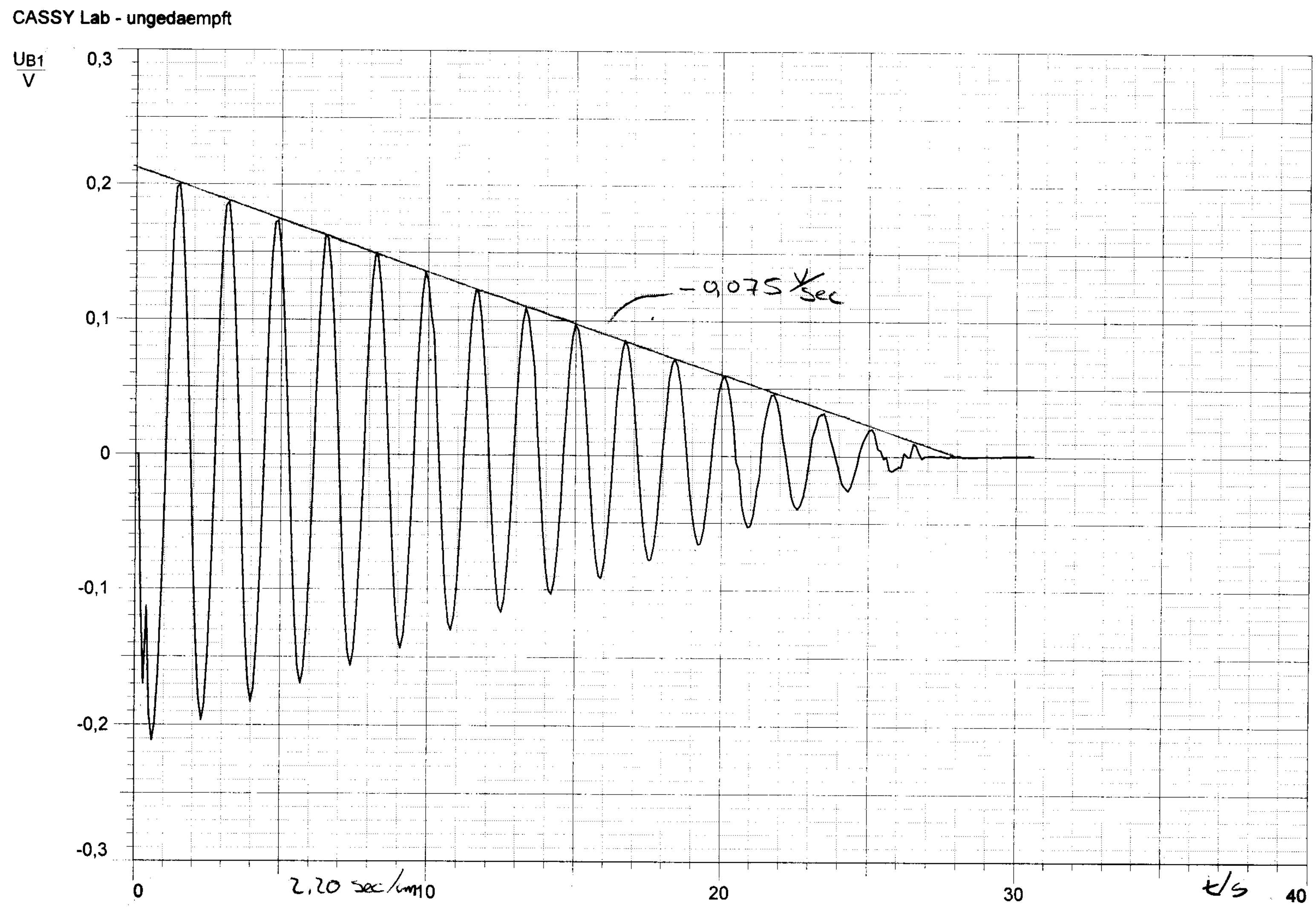
at high Ω ($\Omega_V = 13V$) $\approx 180°$ phase shift

Assignment 4: see printout

Analysis

"Undamped" oscillation:

As a controll measurement, we tried the undamped and undriven oscillator. It appears that there is a constant friction reducing the amplitude. This might have a further effect on other measurements. A linear approximation of that friction shows that there is about $0{,}075\,\frac{V}{sec}$ lost in the amplitude



We could correct the amplitude as $A_{corr} = A \cdot (1 + 0{,}075\,\frac{V}{s} \cdot t)$

In the following measurement of the driven oscillator, however, that would not be a valid calculation because of the constantly acting external force. Instead, the friction will show as a systematic error, the measured amplitudes will be slightly lower than their real value. The exact error would be hard to determine, but it probably is in the order of several percent.

~~Assignment 1~~:

Also, in the experiment with the damped oscillator, this linear correction cannot be applied without destroying the exponential decrease, as we find by trial. We are not completely sure how to correctly include the friction and will ignore it for now, bearing in mind that it should at least give a systematic error.

Assignment 1:

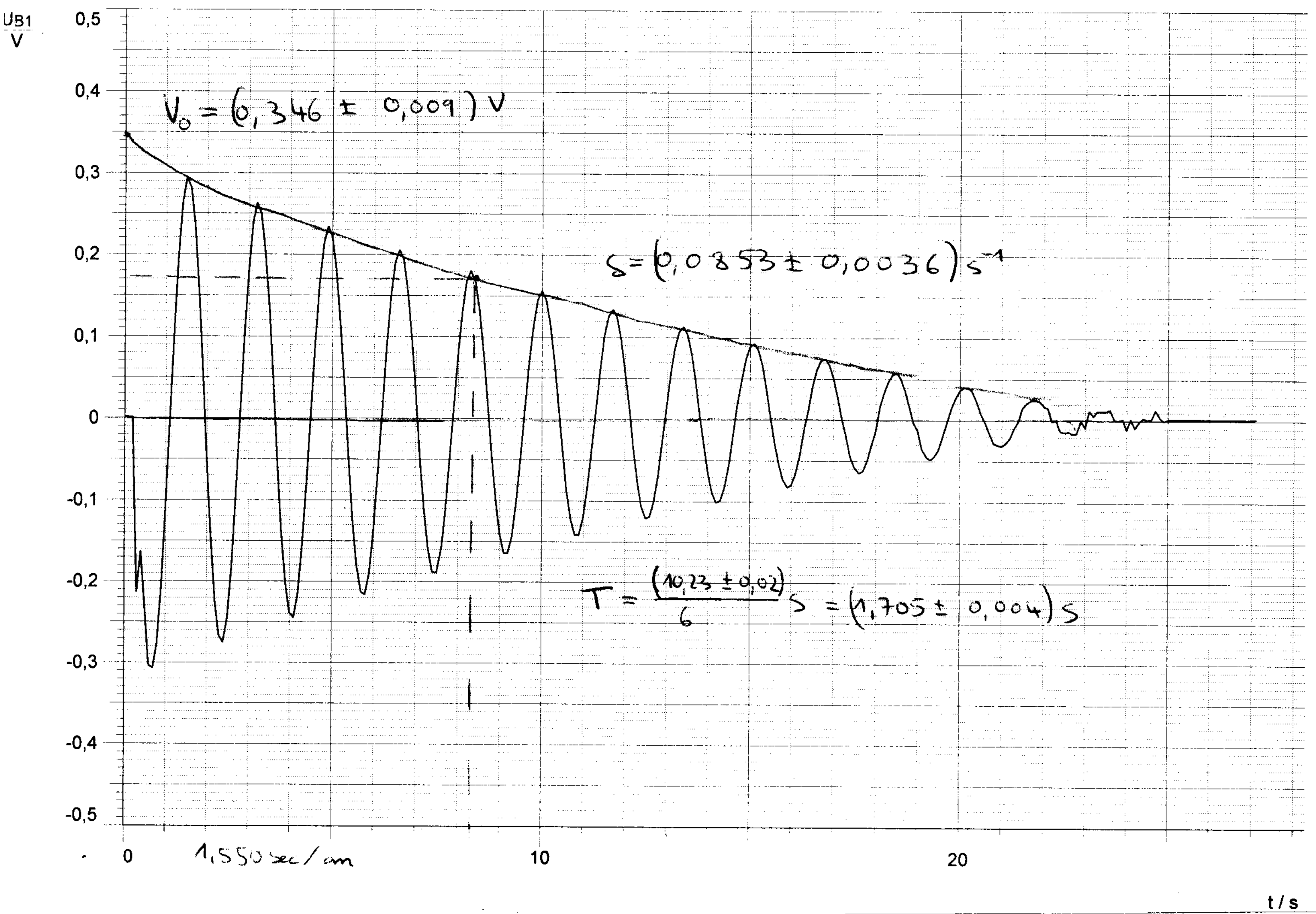
In this assignment, we worked with three different damping levels. For each measurement, we can read δ, the ~~damping~~ constant of decay ~~coefficient~~, from the logscale plot and the time of one period from the normal graph.

From this information, we can calculate ω_0 (frequency of the undamped oscillator) as follows:

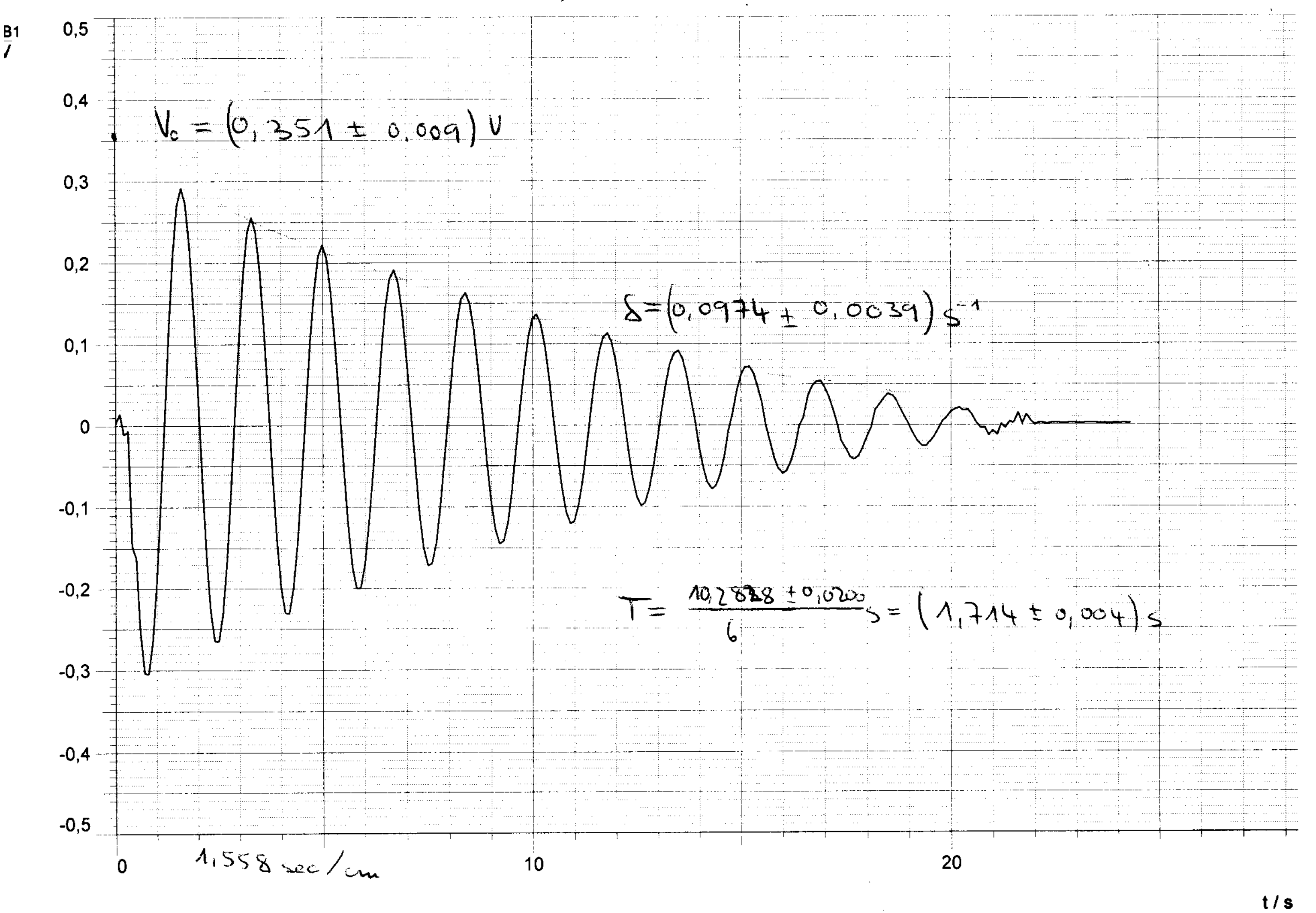
$$\omega_1 = \sqrt{\underbrace{\frac{D}{m}}_{\omega_0^2} - \underbrace{\frac{k^2}{4m^2}}_{\delta^2}} \Leftrightarrow \omega_0^2 = \omega_1^2 + \delta^2$$

Theoretically, we could calculate $k = \delta \cdot 2m$, but since we have neither m nor D we have to use δ as a description for k.

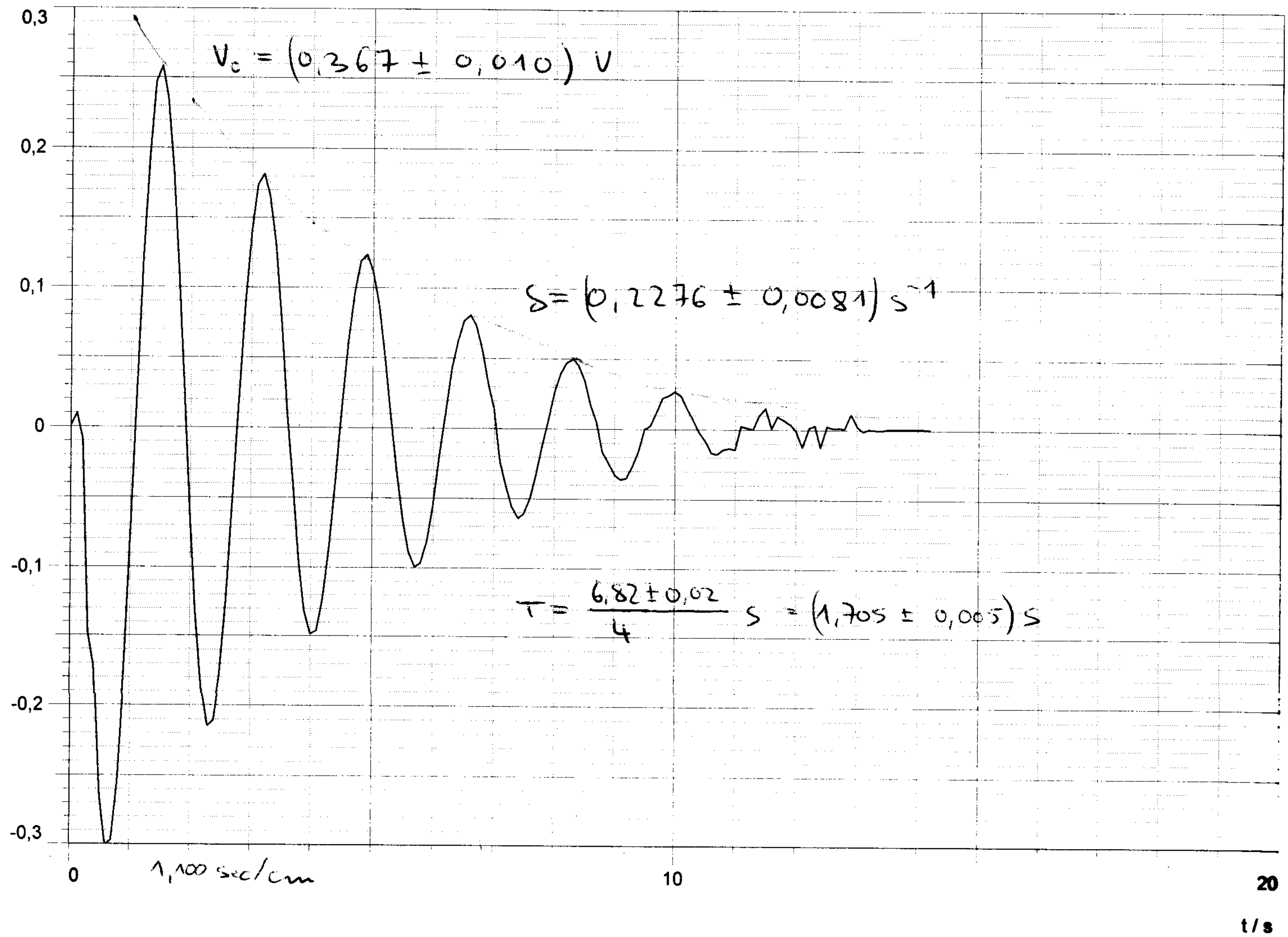
CASSY Lab - damped level 1 - displacement 8



ASSY Lab - damped level 2 - displacement 8



CASSY Lab - damped level 3 - displacement 8



For the eigen frequencies, we get:

level 1: $\omega_0 = 3{,}686$ Hz

level 2: $\omega_0 = 3{,}667$ Hz

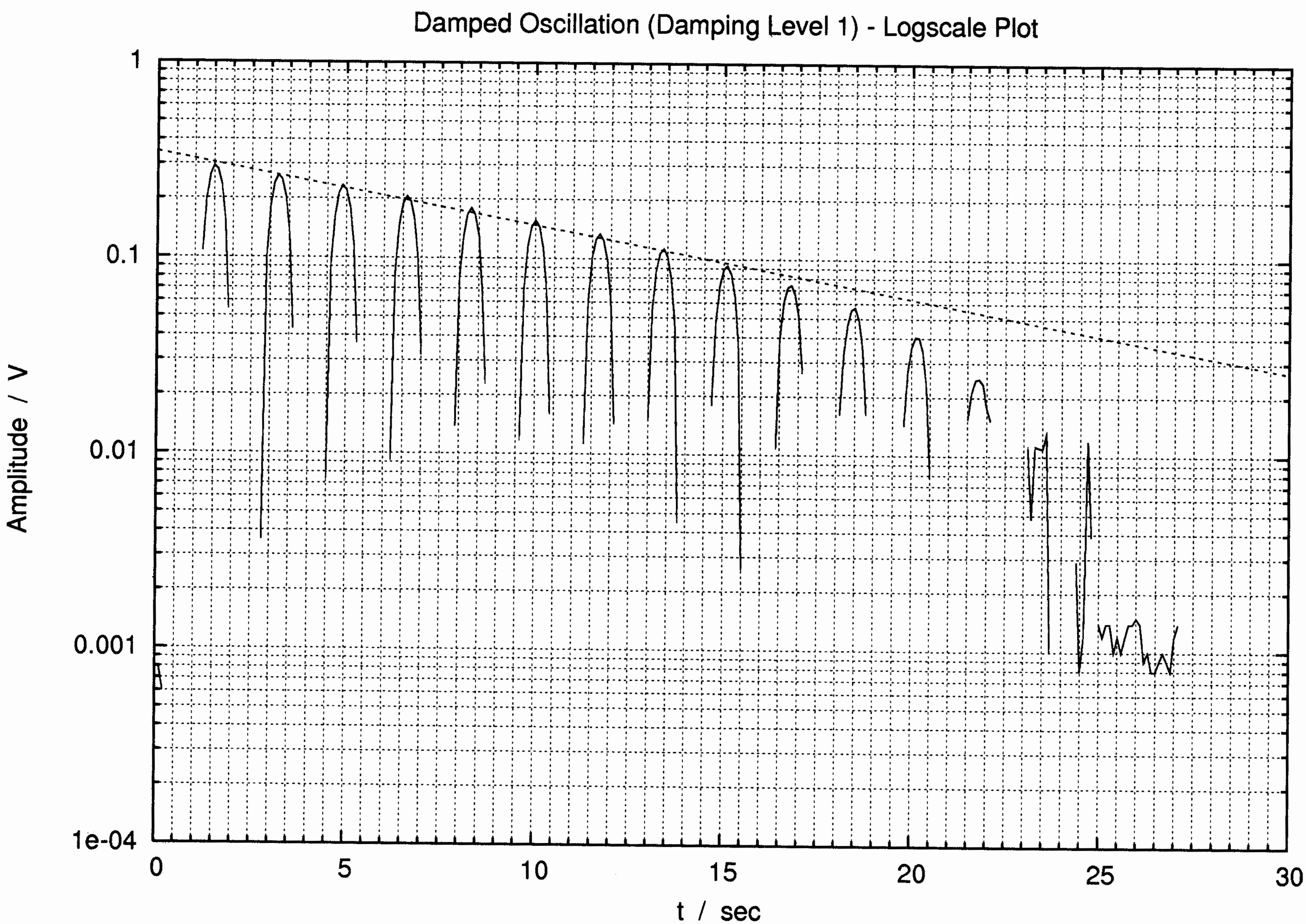
level 3: $\omega_0 = 3{,}692$ Hz

the error on these values is < 1%

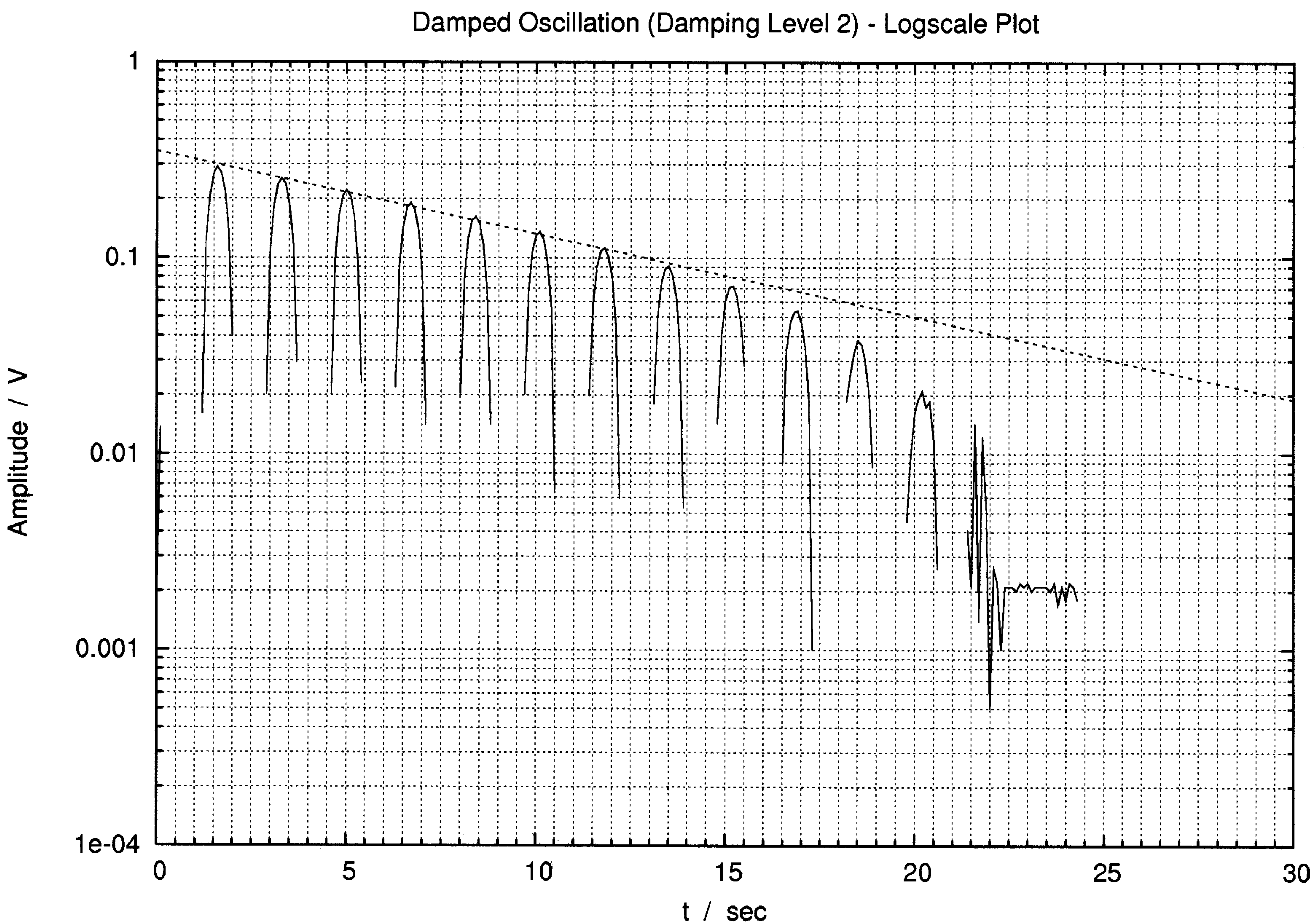
In theory, the values should be identical, the discrepancy is probably due to systematic errors (like the linear friction influence)

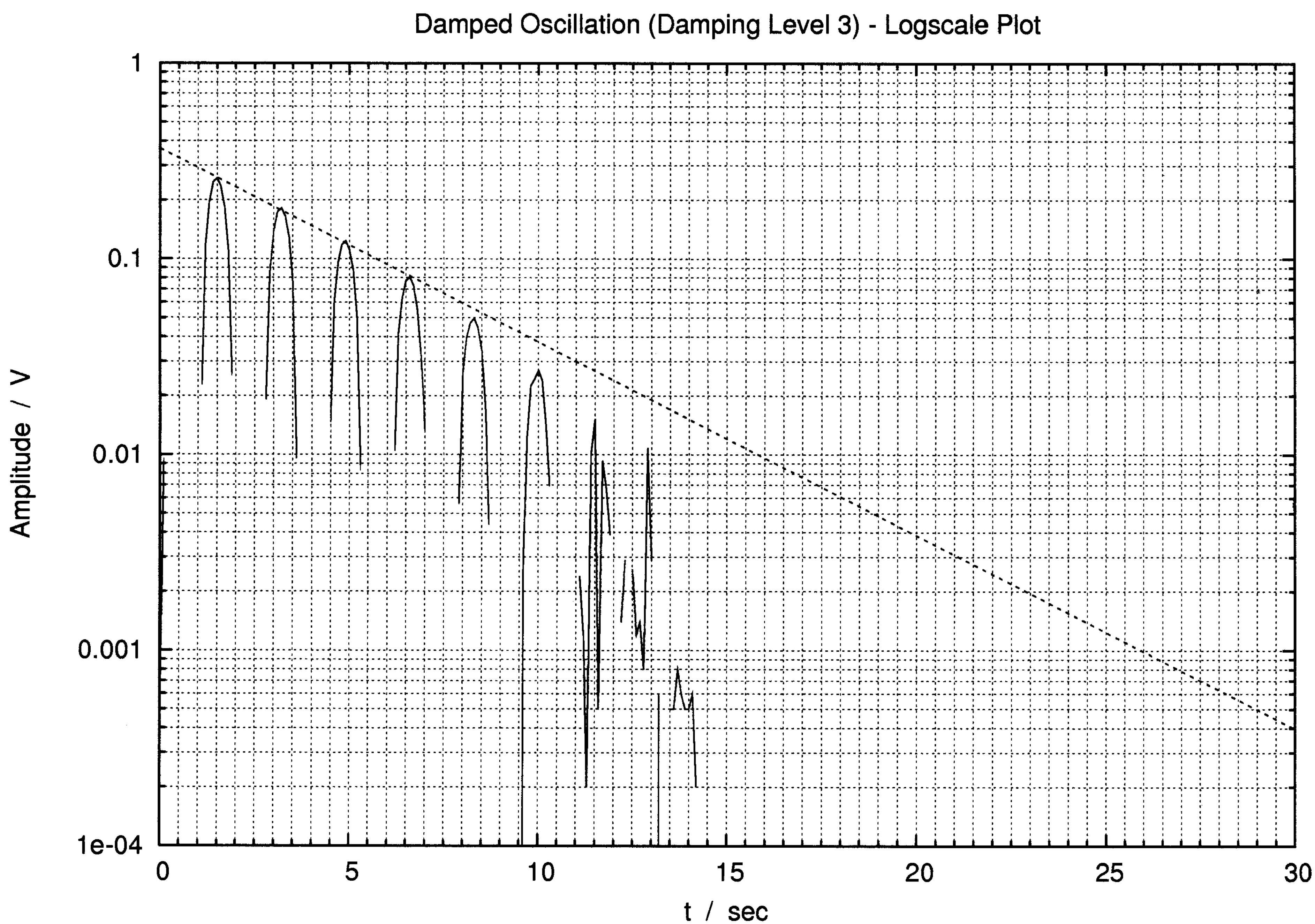
For comparison, the value read from the undamped oscillation is $\omega_0 \approx 3{,}689$ Hz

~~But~~ the exponential decay of your frequency depends of your damping level.



Damped Oscillation (Damping Level 1) - Logscale Plot
Amplitude / V
1
0.1
0.01
0.001
1e-04
0
5
10
15
20
25
30
t / sec


Damped Oscillation (Damping Level 2) - Logscale Plot
1
0.1
0.01
0.001
1e-04
Amplitude / V
0
5
10
15
20
25
30
t / sec


Damped Oscillation (Damping Level 3) - Logscale Plot
Amplitude / V
1
0.1
0.01
0.001
1e-04
0
5
10
15
20
25
30
t / sec

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*******************************************************************************
Tue Mar 22 17:28:16 2005


FIT:    data read from "damped1_orig.csv" every :::1::1
        #datapoints = 11
        residuals are weighted equally (unit weight)

After 6 iterations the fit converged.
final sum of squares of residuals : 0.000731234
rel. change during last iteration : -4.37473e-07

degrees of freedom (ndf) : 9
rms of residuals      (stdfit) = sqrt(WSSR/ndf)      : 0.00901378
variance of residuals (reduced chisquare) = WSSR/ndf : 8.12483e-05

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

A1              = 0.3462           +/- 0.00877      (2.533%)
k1              = 0.085362         +/- 0.003532     (4.137%)


correlation matrix of the fit parameters:

               A1     k1
A1              1.000
k1              0.801  1.000


*******************************************************************************
FIT:    data read from "damped2_orig.csv" every :::1::1
        #datapoints = 10
        residuals are weighted equally (unit weight)

After 6 iterations the fit converged.
final sum of squares of residuals : 0.000528404
rel. change during last iteration : -5.82171e-07

degrees of freedom (ndf) : 8
rms of residuals      (stdfit) = sqrt(WSSR/ndf)      : 0.00812715
variance of residuals (reduced chisquare) = WSSR/ndf : 6.60505e-05

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

A2              = 0.350893         +/- 0.00874      (2.491%)
k2              = 0.0973979        +/- 0.003813     (3.915%)


correlation matrix of the fit parameters:

               A2     k2
A2              1.000
k2              0.809  1.000


*******************************************************************************
FIT:    data read from "damped3_orig.csv" every :::1::1
        #datapoints = 5
        residuals are weighted equally (unit weight)

After 7 iterations the fit converged.
final sum of squares of residuals : 7.16525e-05
rel. change during last iteration : -1.29311e-07

degrees of freedom (ndf) : 3
rms of residuals      (stdfit) = sqrt(WSSR/ndf)      : 0.00488714
variance of residuals (reduced chisquare) = WSSR/ndf : 2.38842e-05

Final set of parameters            Asymptotic Standard Error
=======================            ==========================

A3              = 0.366875         +/- 0.009612     (2.62%)
k3              = 0.227618         +/- 0.008009     (3.519%)


correlation matrix of the fit parameters:

               A3     k3
A3              1.000
k3              0.848  1.000
```

In comparison, the three damped oscillations behave as exspected. The amplitude decreases exponentially, a higher damping level results in a higher decay constant.

Assignment 2:

We can get the eigen frequency directly from the graph of the displacement measured at resonance. We get $\omega_0 = 3{,}738$ Hz without any significant reading error (see next page)

Theoretically, the resonance curve (amplitude in dependancy of external frequency Ω) should also provide a measurement of ω_0, but various problems make this data unusable. Firstly, the resolution around the peak is very low for exact values and secondly, the attempt to convert the voltage at the motor to a frequency Ω was unsuccessful. The data collected for this purpose does not fit any linear law. If we try to do this anyway, we can only say that

$\Omega = (0{,}345 \pm 0{,}109) \cdot V$ (values calculated with gnuplot)

We can then read from the graph

$\omega_0 = (3{,}754 \pm 1{,}186)$ Hz

Likewise, δ should be readable from the graph. The same problems apply. Also, the curve is not symmetric around ω_0 at the level of $\frac{V_0}{\sqrt{2}}$, where we could read of δ.

The average value is $\delta = 0{,}0328 \pm 0{,}0053$,
which is still not anywhere near the exspected value.
In addition to the mentioned problems, there are also systematic errors.
Qualitatively, however, the resonance curve is exactly as we expect, with the sharp peak at the resonance frequency and the asymmetric development for left and right from that frequency

Assignment 3:

The observations follow our expectations. At low driving frequencies, there is enough time for the spring to follow the motor's movement. With increasing frequency, the oscillator's inertia will create a phase shift. At resonance, the phase shift will be exactly 90°, which makes sense, as one can add a maximum of energy when exerting force at the moment when the pendulum goes through its rest position (just like on a swing)
Finally, at ~~the~~ very high frequencies, the pendulum's inertia is too high to follow the motor, the phase shift becomes 180°

Assignment 4:

The initial phases behave like exspected. The first graph (10,8V) is barely below the resonance frequency. We can see that the amplitude reaches a higher level at first, and then stabilizes on the lower level. The terminal phase, when the motor is switched off, behaves like

→

(Assg. 4, cont.)

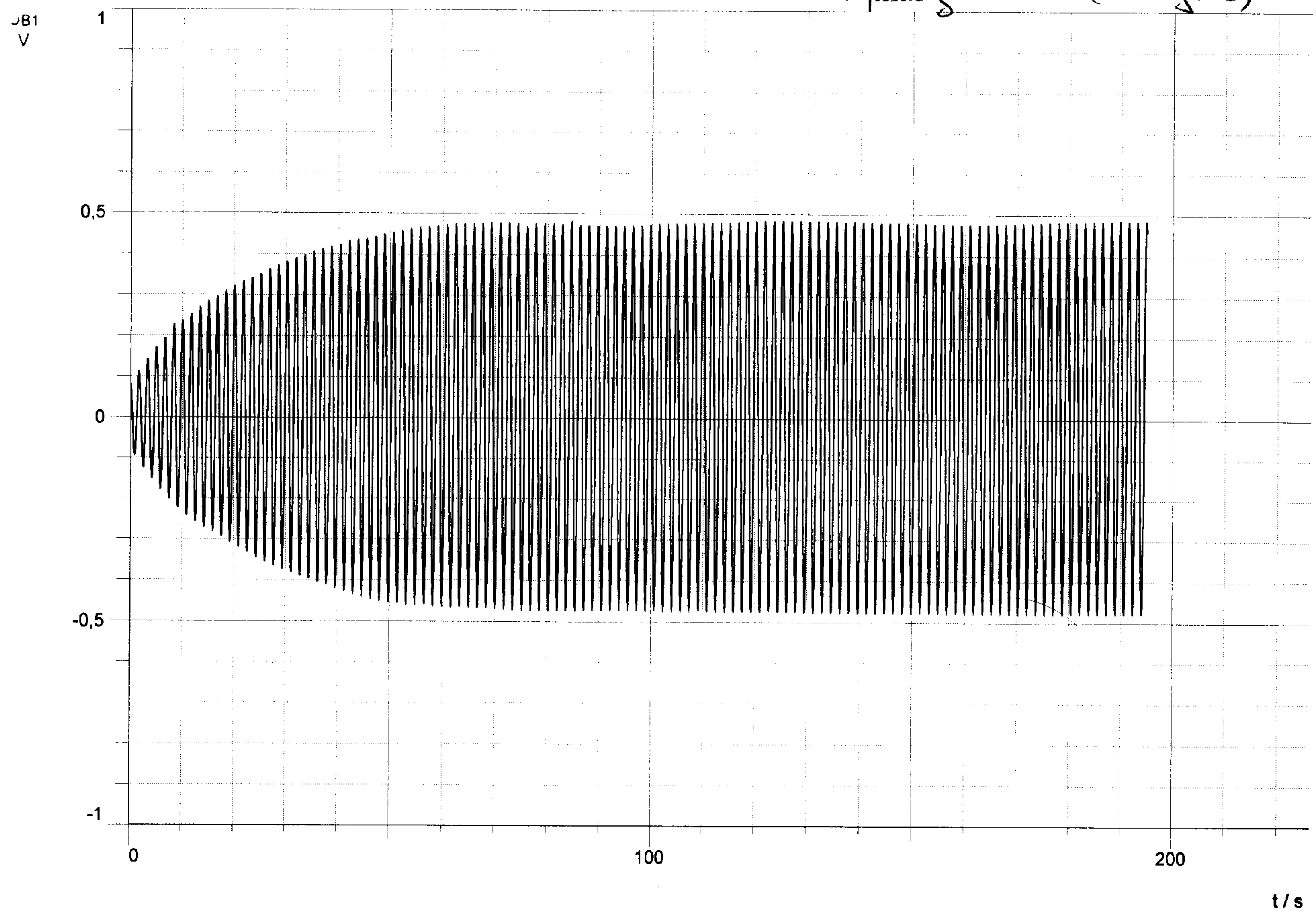
the undamped oscillator at the beginning (in the first experiment)

At the resonance frequency, the amplitude directly goes to its maximum and stays there (almost without reaching a higher value first)

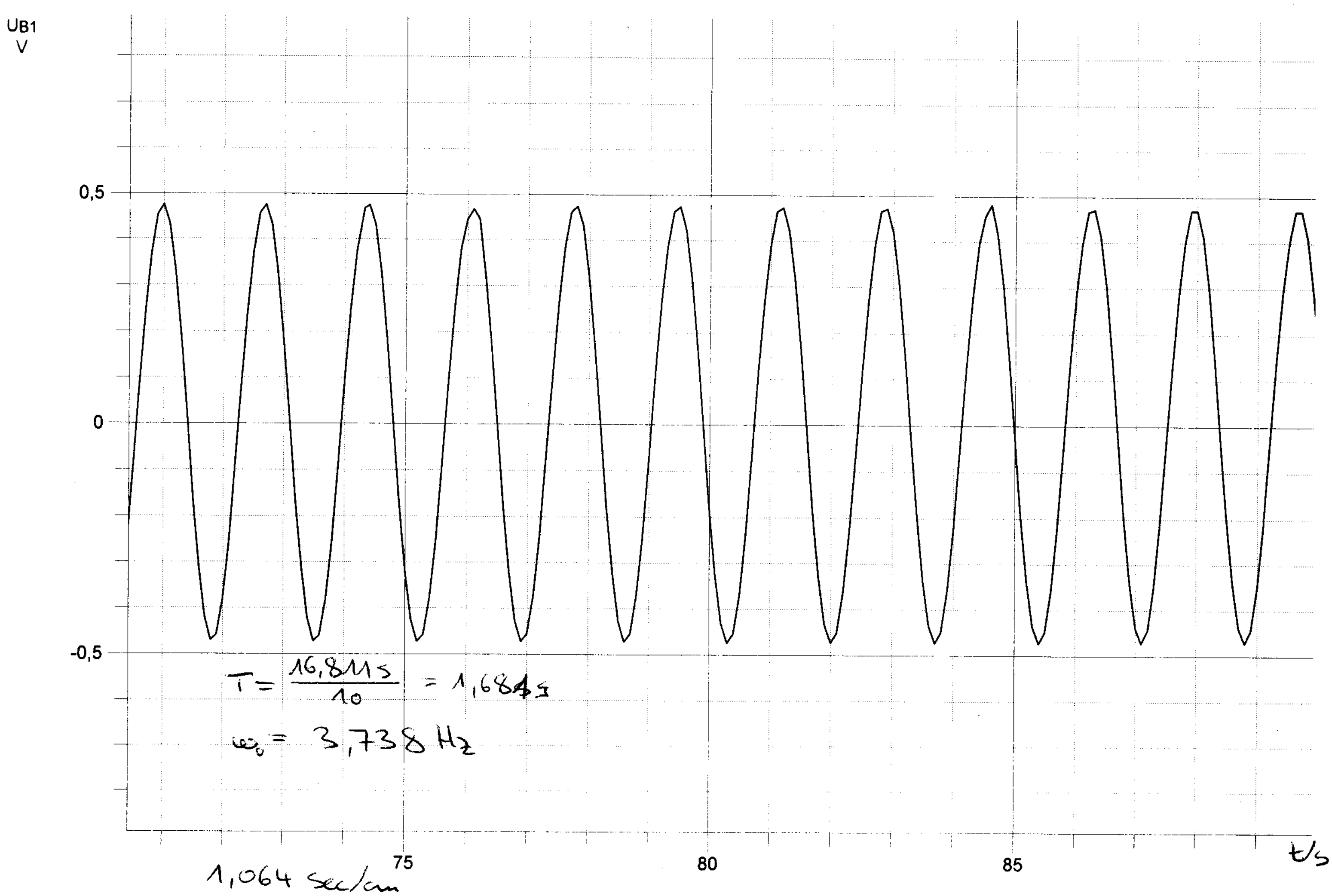
At a frequency above resonance, we clearly see the characteristic beats. The amplitude also is much lower, as ~~we~~ we are further away from the resonance

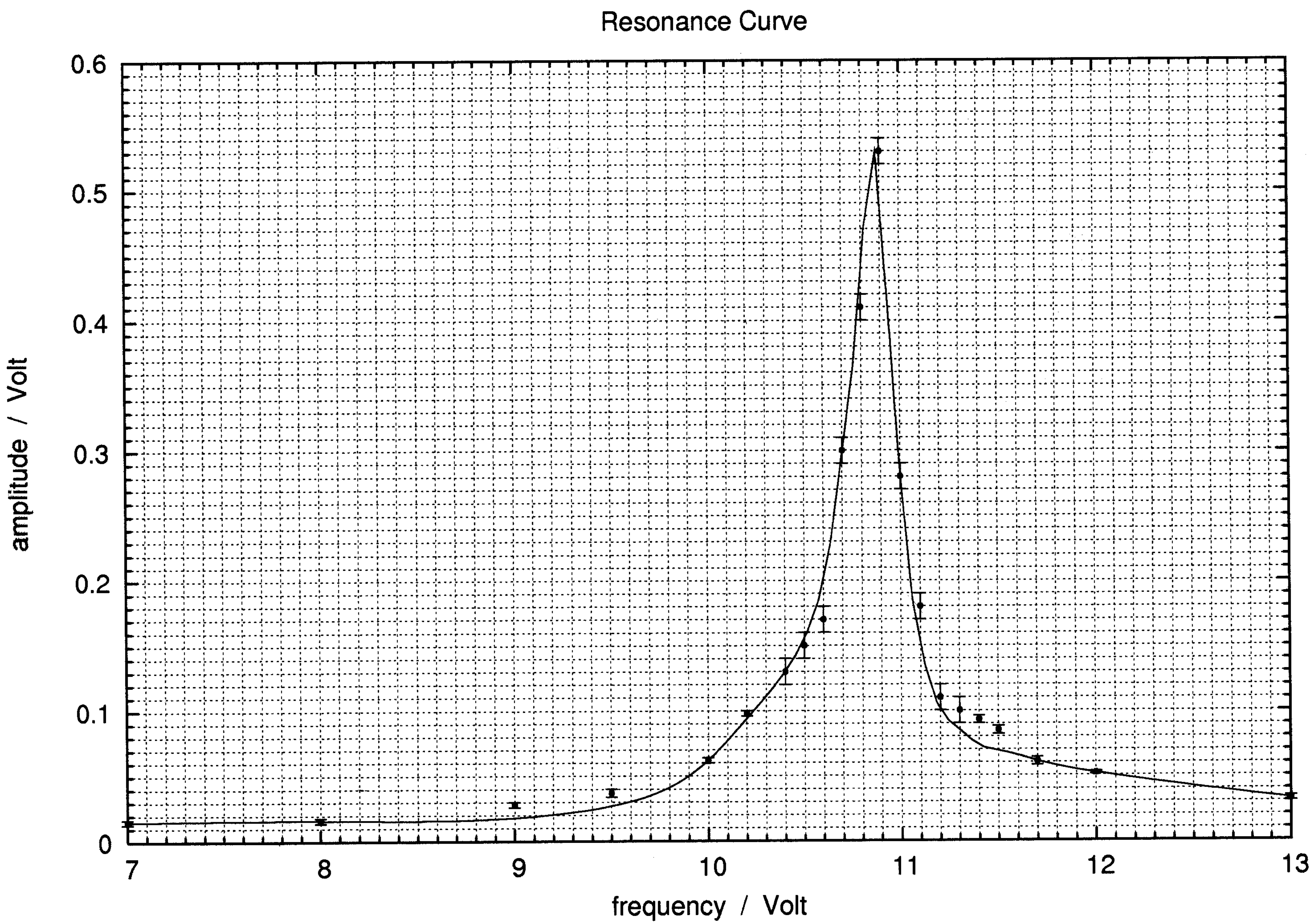
Measurement at Resonance Frequency (Assg. 2)

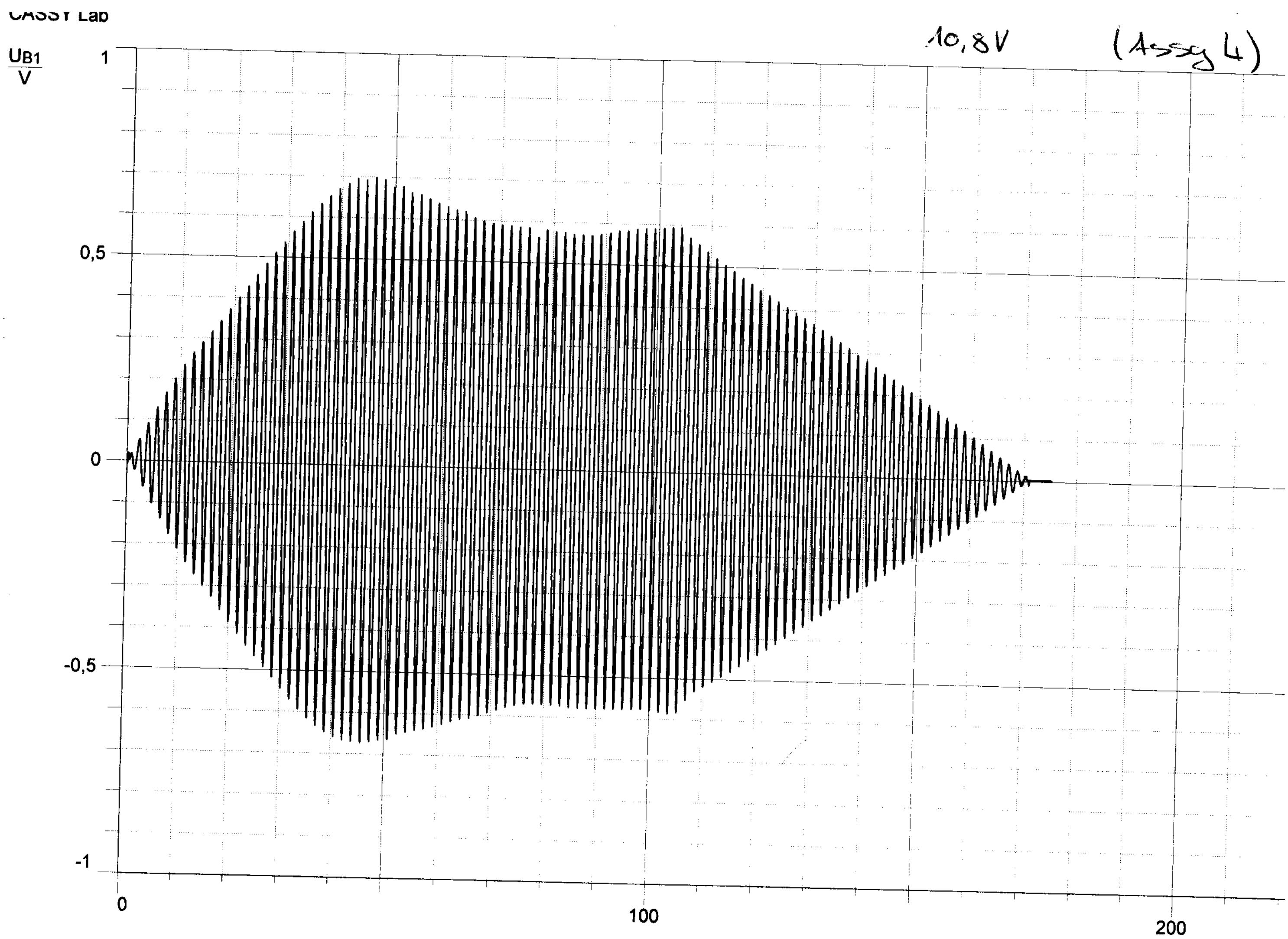
CASSY Lab - forced oscillation - damping level 1

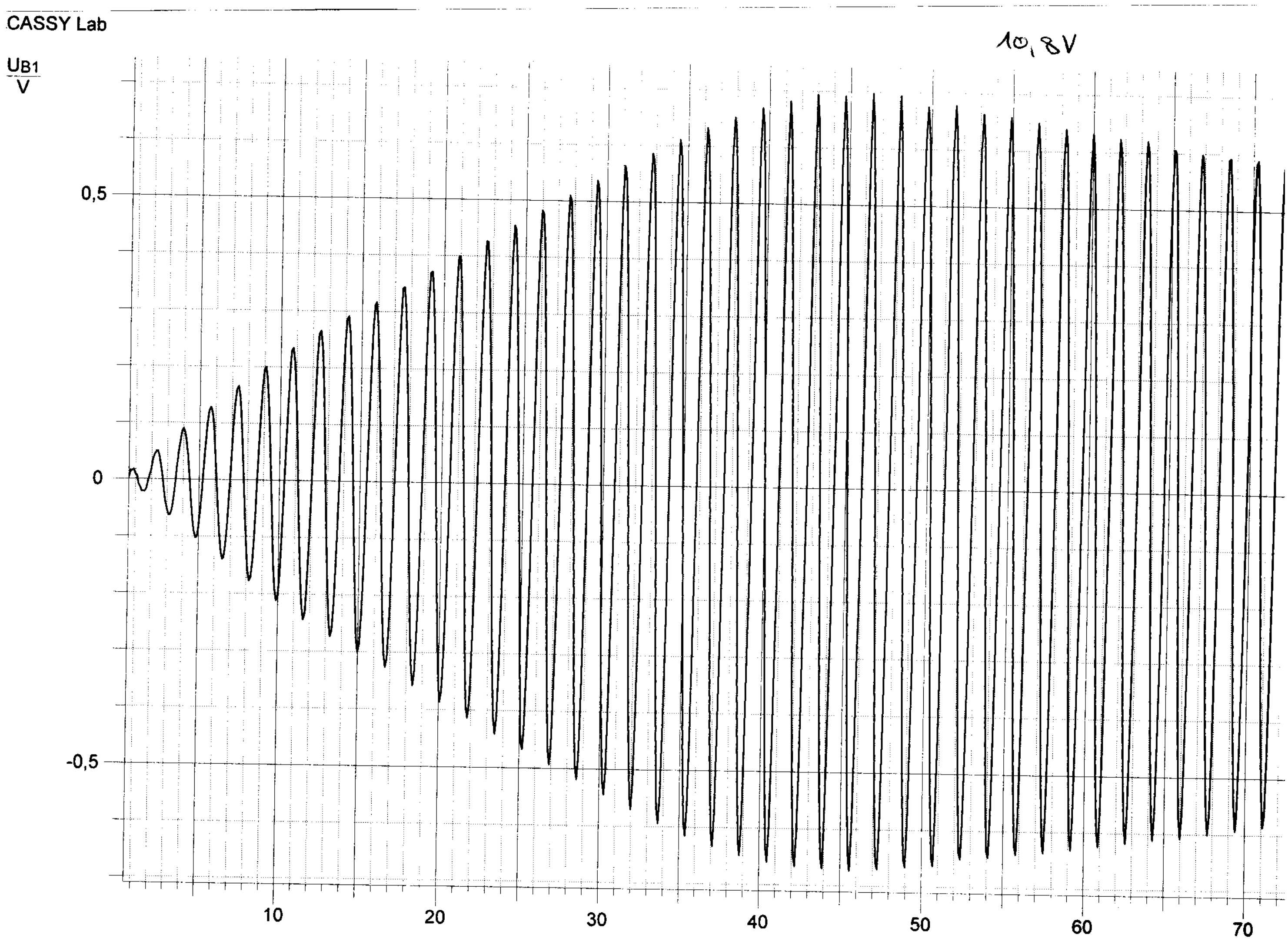


CASSY Lab - forced oscillation - damping level 1

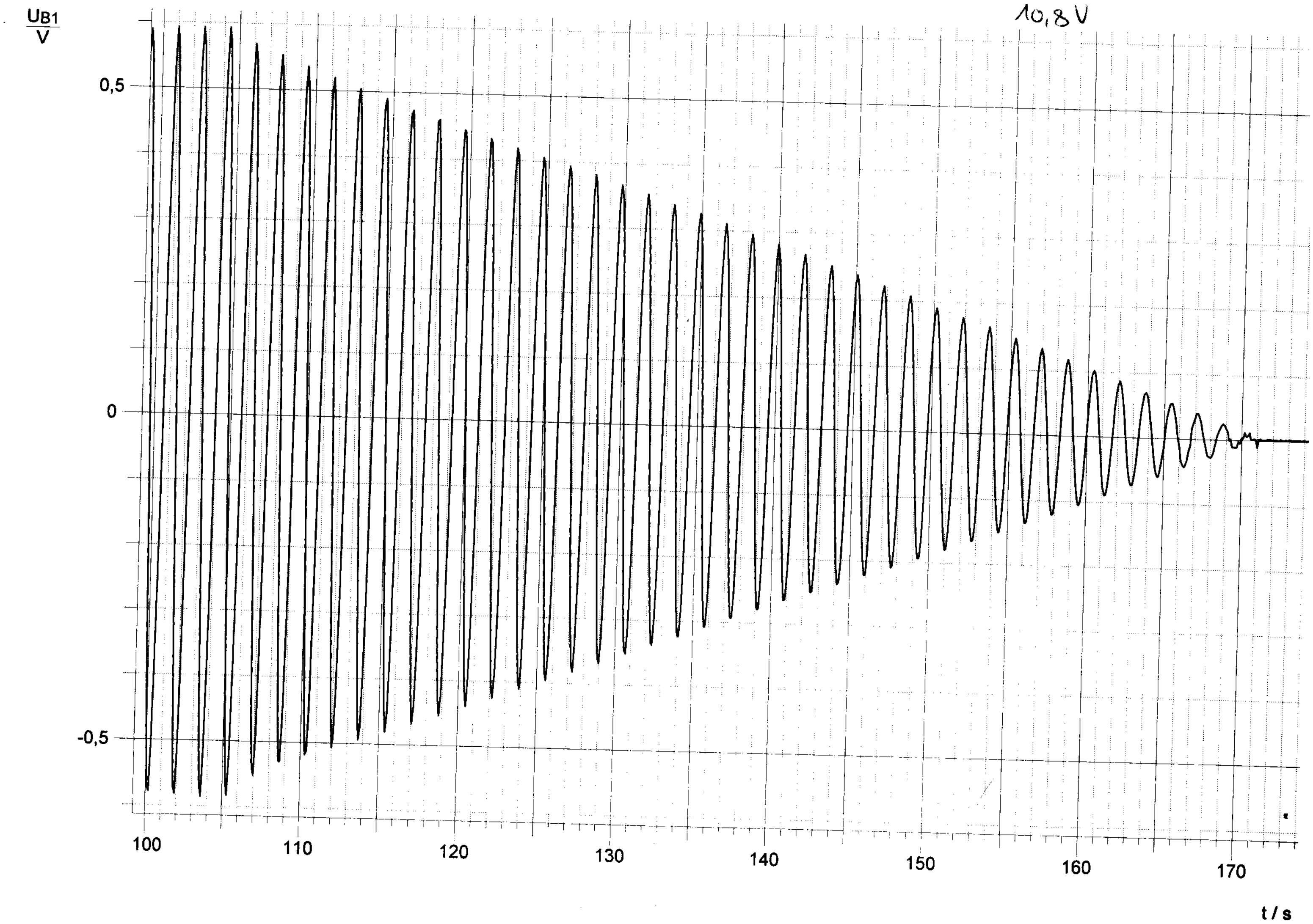


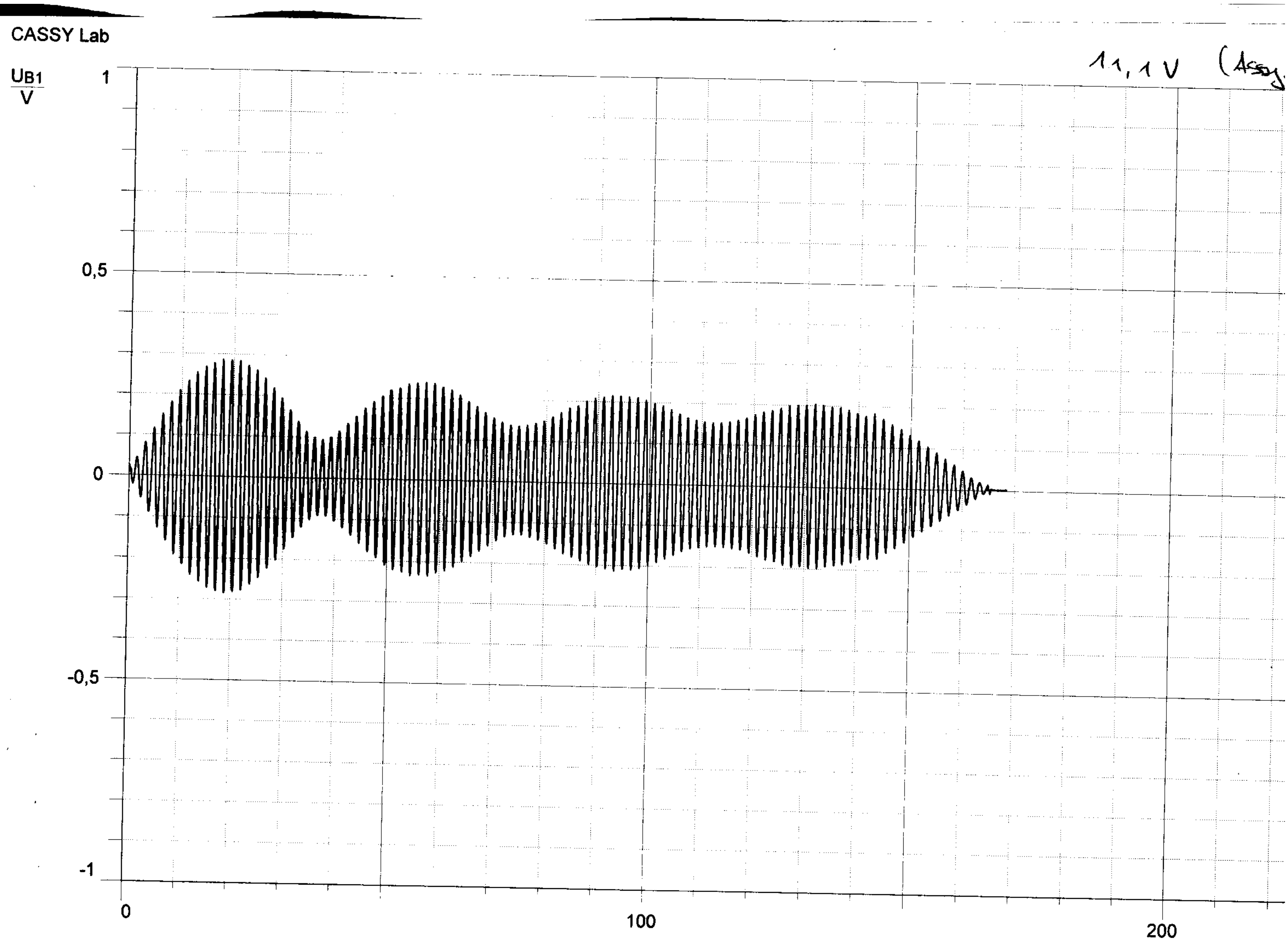

Resonance Curve
amplitude / Volt
0.6
0.5
0.4
0.3
0.2
0.1
0
7
8
9
10
11
12
13
frequency / Volt

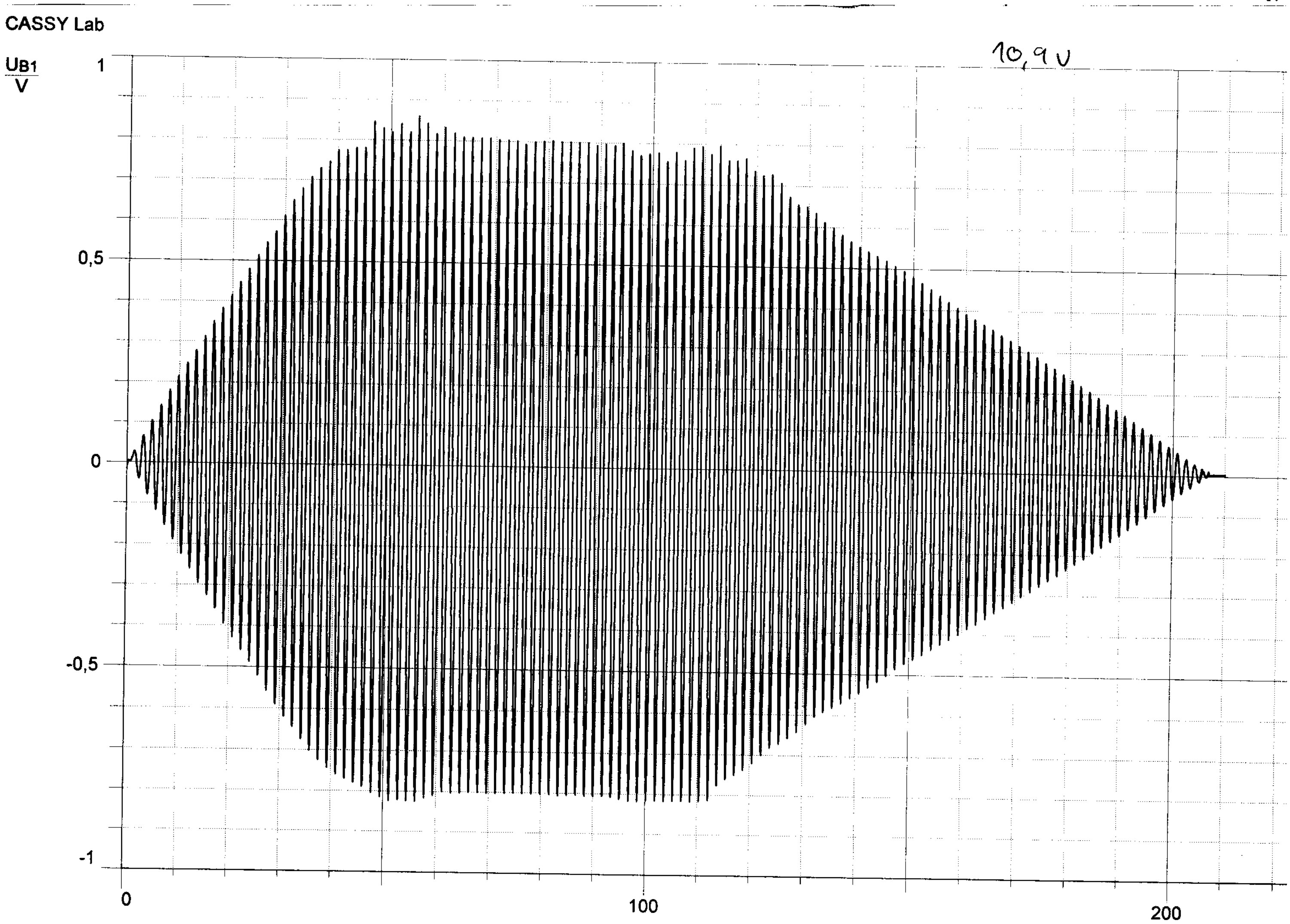

CASSY Lab
10,8V
(Assy 4)
$\frac{U_{B1}}{V}$
1
0,5
0
-0,5
-1
0
100
200


CASSY Lab
10,8V
$\frac{U_{B1}}{V}$
0,5
0
-0,5
10
20
30
40
50
60
70

CASSY Lab




CASSY Lab
UB1 / V
11,1 V (Cassy
1
0,5
0
-0,5
-1
0
100
200
'S


t /
CASSY Lab
UB1 / V
10,9 V
1
0,5
0
-0,5
-1
0
100
200

Conclusion

Assignment 1 allowed to measure the eigen frequency and the decay constant rather accurately. There are no literature values for comparison, but the measured values seem reasonable and are in agreement with each other. The exponential decay of the amplitude that we expected was clearly visible.

Assignment 2 gave a resonance curve as exspected, but was a failure for quantitative values, as we already discussed in the analysis. The low accuracy of the motor and the impossibility to set the frequency directly did not allow any useful measurement beyond qualitative examination

Assignment 3 was in full correspondance wit our exspectation.

Assignment 4 also qualitatively behaved according to the theory. Different effect during the initial phase were clearly visible

Good!

5.0 23.5.97